Replaces Pages 280-285,
Jehle, G. and P. Reny, *Advanced Microeconomic Theory, 2nd. Ed.*

7.2.3 INCOMPLETE INFORMATION

Although a large variety of situations can be modeled as stategic form games, our analysis of these games so far seems to be subject to a rather important limitation. Until now, when we've considered iterative strict or weak dominance, or Nash equilibrium as our method of solving a game, we've always assumed that every player is perfectly informed of the payoffs of all other players. Otherwise, the players could not have carried out the calculations necessary for deriving their optimal strategies.

But many real-life situations involve substantial doses of incomplete information about the opponents' payoffs. Consider, for instance, two firms competing for profits in the same market. It is very likely that one or both of them is imperfectly informed about the other's costs of production. How are we to analyze such a situation? The idea is to add to it one more ingredient so that it becomes a strategic form game. We'll then be able to apply any of the various solution methods that we've developed so far. These ideas were pioneered in Harsanyi (1967–1968).

The additional ingredient is a specification of firm 1's beliefs, in the form of a probability distribution, about firm 2's cost. For example, we might specify that firm 1 believes that it is equally likely that firm 2 is a high- or low-cost firm. Before getting too far ahead, it is worthwhile to formalize our thoughts up to now.

Consider the following class of strategic situations in which information is incomplete. As usual, there are a finite number of players $i = 1, \dots, N$, and a pure strategy set, S_i, for each of them. In addition, however, there may be uncertainty regarding the preferences of some of them. To capture this, we introduce for each player i a finite set, T_i, of possible "types" that player might be. We allow a player's payoff to depend as usual on the chosen joint pure strategy, but also on his own type as well as on the types of the others. That is, player i's payoff function u_i maps $S \times T$ into R, where $T = \times_{i=1}^{N} T_i$, and S is the set of joint pure strategies.

Finally, we introduce the extra ingredient that will allow us to use the solutions we've developed in previous sections. Let p be a probability distribution over the set T of player types such that $p(t) > 0$ for every $t \in T$. The probability distribution p is often referred to as a **common prior** because it is the common probability distribution that all players use to assess the probability that the players' types assume any particular value.

The common prior assumption can be understood in at least two ways. The first is that p is simply an objective empirical distribution over the players' types, one that has been borne out through many past observations. The second is that the common prior assumption reflects the idea that differences in beliefs arise only from differences in information. Consequently, before the players are aware of their own types—and are therefore in a symmetric position—each player's beliefs about the vector of player types must be identical, and equal to p.

If player i is of type $\bar{t}_i \in T_i$, then his uncertainty regarding the types of the others will be captured by the **conditional probability** distribution on T_{-i} derived from p given $\bar{t}_i$. That is, when player i's type is $\bar{t}_i$, he assigns probability

$$p(t_{-i} \mid \bar{t}_i) = p(\bar{t}_i, t_{-i}) \Big/ \sum_{t_{-i} \in T_{-i}} p(\bar{t}_i, t_{-i})$$

to the possibility that the others' types are t_{-i}. (This is nothing more than Bayes' rule, which can be found in any good probability text. See also section 7.3.7.)

Before we describe how to analyze such a situation, we place all of these elements together.

DEFINITION 7.1 *Game of Incomplete Information*

A game of incomplete information is a tuple $G = (p, T_i, S_i, u_i)_{i=1}^N$, where p is a probability distribution on T giving each member strictly positive probability, and for each player $i = 1, \ldots, N$, the set T_i is finite, and $u_i\colon S \times T \to \mathrm{R}$. If in addition, for each player i, the strategy set S_i is finite, then G is called a finite game of incomplete information.

The question remains: How can we apply our previously developed solutions to incomplete information games? The answer is to reinterpret the game as follows. Consider each type of every player in the game of incomplete information as a *separate player*, and suppose that "Nature," using the probability distribution p, randomly chooses which of these types actually will play the game. In addition, in this reinterpretation, each type of every player must choose his strategy before Nature's random choice is made. As we shall see, this creates a strategic form game in the sense of Definition 7.1 that can be analyzed as before. Moreover, this strategic form game beautifully captures all relevant aspects of the original game of incomplete information. Consider the following example.

EXAMPLE 7.3 Two firms are engaged in Bertrand price competition as in Chapter 4, except that one of them is uncertain about the other's constant marginal cost. Firm 1's marginal cost of production is known, and firm 2's is either high or low, with each possibility being equally likely. There are no fixed costs. Thus, firm 1 has but one type, and firm 2 has two types—high cost and low cost. The two firms each have the same strategy set, namely the set of nonnegative prices. Firm 2's payoff depends on his type, but firm 1's payoff is independent of firm 2's type; it depends only on the chosen prices.

To derive from this game of incomplete information a strategic form game, imagine that there are actually *three* firms rather than two, namely, firm 1, firm 2 with high cost, and firm 2 with low cost. Imagine also that *each* of the three firms must simultaneously choose a price and that after doing so, one of the firm 2's is chosen (with probability 1/2 on each) to compete with firm 1 at the previously chosen prices. Although this may appear to be rather fanciful, some thought will convince you that it beautifully captures all the relevant strategic features of the original situation. In particular, firm 1 must choose its price without knowing whether its competitor has high or low costs. Moreover, firm 1 understands that the competitor's price may differ according to its costs. □

In general then, we wish to associate with each game of incomplete information a strategic form game capturing its essential features. This is done as follows.

DEFINITION 7.2 *The Associated Strategic Form Game*

Let $G = (p, T_i, S_i, u_i)_{i=1}^N$ be a game of incomplete information, and let $G^* = (R_j, v_j)_{j \in J}$ be the strategic form game whose player set, J, is the set of all indices, j, of the form $j = (i, t_i)$, where $t_i \in T_i$ and $i = 1, 2, \ldots N$; and where player $j = (i, t_i)$'s strategy set and payoff function are defined, respectively, by

$$R_j \equiv S_i, \qquad \textit{and}$$
$$v_j(r) \equiv \sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) u_i(r_j, (r_{(k,t_k)})_{k \neq i}, t_i, t_{-i}),$$

where $r \in \times_{l \in J} R_l$. We then say that G^* is the strategic form game associated with the incomplete information game G.

By associating with each game of incomplete information a well-chosen strategic form game, we have reduced the study of games of incomplete information to the study of games with complete information, that is, to the study of strategic form games. Consequently, we may apply any of the solutions that we've developed to the associated strategic form game. In particular, we may consider the set of Nash equilibria of the associated game. Indeed, this has proved to be particularly useful.

DEFINITION 7.3 *Bayesian-Nash Equilibrium*

A Bayesian-Nash equilibrium of a game of incomplete information is a Nash equilibrium of the associated strategic form game.

With the tools we've developed up to now, it is straightforward to deal with the question of existence of Bayesian-Nash equilibrium.

THEOREM 7.1 *Existence of Bayesian-Nash Equilibrium*

Every finite game of incomplete information possesses at least one Bayesian-Nash equi-librium.

$1 \setminus 2l$	$p_l = 6$	$p_l = 4$	$p_l = 1$
$p_1 = 6$	6, 5	0, 12	0, 0
$p_1 = 4$	16, 0	8, 6	0, 0
$p_1 = 1$	7, 0	7, 0	7, 0

$1 \setminus 2h$	$p_h = 6$	$p_h = 4$	$p_h = 1$
$p_1 = 6$	6, 2	0, 0	0, -21
$p_1 = 4$	16, 0	16, 0	0, -21
$p_1 = 1$	7, 0	7, 0	7, 0

Figure 7.7. A Bertrand-competition incomplete information game.

Proof: By Definition 7.12, it suffices to show that the associated strategic form game possesses a Nash equilibrium. Because the strategic form game associated with a finite game of incomplete information is itself finite, we may apply Theorem 7.2 to conclude that the associated strategic form game possesses a Nash equilibrium. ■

EXAMPLE 7.4 To see these ideas at work, let's consider in more detail the two firms discussed in Example 7.2. Suppose that firm 1's marginal cost of production is zero. Also, suppose that firm 2's marginal cost is either 1 or 4, and that each of these "types" of firm 2 occur with probability 1/2. If the lowest price charged is p, then market demand is $8 - p$. To keep things simple, we'll suppose that each firm can choose only one of three prices, 1, 4, or 6. The payoffs to the firms are described in Fig. 7.7. Firm 1's payoff is always the first number in any pair, and firm 2's payoff when his costs are low (high) are given by the second number in the entries of the matrix on the left (right).

In keeping with the Bertrand-competition nature of the problem, we have instituted the following convention in determining payoffs when the firms choose the same price. If both firms' costs are strictly less than the common price, then the market is split evenly between them. Otherwise, firm 1 captures the entire market at the common price. The latter uneven split reflects the idea that if the common price is above only firm 1's cost, firm 1 could capture the entire market by lowering his price slightly (which, if we let him, he could do and still more than cover his costs), whereas firm 2 would not lower his price (even if we let him) because this would result in losses.

We have now described the game of incomplete information. The associated strategic form game is one in which there are three players: firm 1, firm $2l$ (low cost), and firm $2h$ (high cost). Each has the same pure strategy set, namely, the set of prices $\{1, 4, 6\}$. Let p_1, p_l, p_h denote the price chosen by firms 1, $2l$, and $2h$, respectively.

Firm 1 chooses $p_1 = 6$

$2l \setminus 2h$	$p_h = 6$	$p_h = 4$	$p_h = 1$
$p_l = 6$	6, 5, 2	3, 5, 0	3, 5, -21
$p_l = 4$	3, 12, 2	0, 12, 0	0, 12, -21
$p_l = 1$	3, 0, 2	0, 0, 0	0, 0, -21

Firm 1 chooses $p_1 = 4$

$2l \setminus 2h$	$p_h = 6$	$p_h = 4$	$p_h = 1$
$p_l = 6$	16, 0, 0	16, 0, 0	8, 0, -21
$p_l = 4$	12, 6, 0	12, 6, 0	4, 6, -21
$p_l = 1$	8, 0, 0	8, 0, 0	0, 0, -21

Firm 1 chooses $p_1 = 1$

$2l \setminus 2h$	$p_h = 6$	$p_h = 4$	$p_h = 1$
$p_l = 6$	7, 0, 0	7, 0, 0	7, 0, 0
$p_l = 4$	7, 0, 0	7, 0, 0	7, 0, 0
$p_l = 1$	7, 0, 0	7, 0, 0	7, 0, 0

Figure 7.8. The associated strategic form game.

Fig. 7.8 depicts this strategic form game. As there are three players, firm 1's choice of price determines the matrix, and firms $2l$ and $2h$'s prices determine the row and column, respectively, of the chosen matrix. For example, according to Fig. 7.8, if firm 1 chooses $p_1 = 4$, firm $2l$ $p_l = 4$, and firm $2h$ $p_h = 4$, their payoffs would be 12, 6, and 0, respectively.

According to Definition 7.11, the payoffs in the strategic form game of Fig. 7.8 for firms $2l$ and $2h$ can be obtained by simply reading them off of the matrices from Fig. 7.7. This is because there is only one "type" of firm 1. For example, according to Fig. 7.7, if firm $2l$ is chosen by Nature and chooses $p_l = 6$, then it receives a payoff of 5 if firm 1 chooses $p_1 = 6$. Note that this is reflected in the associated game of Fig. 7.8, where firm $2l$'s payoff is 5 when it and firm 1 choose a price of 6 regardless of the price chosen by firm $2h$.

The payoffs to firm 1 in the associated strategic form game of Fig. 7.8 are obtained through the use of the prior distribution on firm 2's costs. For example, consider the strategy in which firm $2l$ chooses $p_l = 1$, firm $2h$ chooses $p_h = 6$, and firm 1 chooses $p_1 = 4$. Now, if firm $2l$ is chosen (by Nature), then according to Fig. 7.7, firm 1's payoff is zero. If firm $2h$ is chosen, then firm 1's payoff is 16. Because each type of firm 2 is chosen with probability $1/2$ according to the prior, firm 1's expected payoff is 8. This is precisely firm 1's payoff corresponding to $p_1 = 4, p_l = 1$, and $p_h = 6$ in Fig. 7.8. One can similarly calculate firm 1's associated strategic form game (expected) payoff given in Fig. 7.8 for all other joint strategy combinations.

To discover a Bayesian-Nash equilibrium of the Bertrand-competition incomplete information game, we must look for a Nash equilibrium of the associated strategic form game of Fig. 7.8.

Finding one Nash equilibrium is particularly easy here. Note that firms $2l$ and $2h$ each have a weakly dominant strategy: choosing a price of 4 is weakly dominant for firm $2l$ and choosing a price of 6 is weakly dominant for firm $2h$. But once we eliminate the other strategies for them, firm 1 then has a strictly dominant strategy, namely, to choose a price of 4. To see this, suppose that $p_l = 4$ and $p_h = 6$. Then according to Fig. 7.8, firm 1's payoff is 3 if he chooses $p_1 = 6$, 12 if he chooses $p_1 = 4$, and 7 if he chooses $p_1 = 1$.

Consequently, there is a pure strategy Bayesian-Nash equilibrium in which two of the three firms choose a price of 4 while the third chooses a price of 6. You are invited to explore the existence of other Bayesian-Nash equilibria of this game in an exercise. Note that in contrast to the case of Bertrand competition with complete information, profits are not driven to zero here. Indeed, only the high-cost firm 2 earns zero profits in the equilibrium described here. $\square$