Solutions to Jehle and Reny (3rd ed.), Chapter 9, 9.6-9.36

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9.6. In the second-price and English auctions, each bidder's bid is $b_i(v_i) = v_i$, the highest bidder j gets the object and pays $c_j(v) = \max\{v_i, \forall i \neq j\}$, and the rest of the bidders pay nothing.

Denote v^{II} the second largest value of v. Consider the following direct selling mechanism $(p_i(v), c_i(v))$: $p_i(v) = 1$ for $v_i > v^{II}$, $p_i(v) = 0$ otherwise, and $c_i(v) = p_i(v)v_i - \int_0^{v_i} p_i(x, v_{-i})dx$ for all i. Clearly, $p_i(v)$ is non-decreasing in v_i, so is $\bar{p}_i(v_i)$, and $\bar{c}_i(v) = \bar{c}_i(0) + \bar{p}_i(v_i)v_i - \int_0^{v_i} \bar{p}_i(x)dx$ with $\bar{c}_i(0) = 0$, hence the mechanism is incentive-compatible. Now, for any given v, $p_j(v) = 1$ for $v_j > v^{II}$, that is, the object is assigned to the individual with the highest bid. And winner pays $c_j(v) = p_j(v)v_j - \int_0^{v_j} p_j(x, v_{-i})dx = v_j - \int_{v^{II}}^{v_j} 1dx = v^{II}$, and the others pay $c_i(v) = 0$, $i \neq j$. The outcome is exactly the same as the second-price sealed-bid or English auction.

Note that the distributions of the bidders' values are not assumed to be independent and symmetric. In fact, the seller doesn't have to know the exact distribution functions.

In the equilibrium of the first-price and Dutch auctions, each bidder bids $b_i(v_i)$, which can be shown non-decreasing in v_i and not grater than v_i, and the winner pays his own bid while the rest pay nothing.

Given v and $b_i(v_i)$, let $b_j(v_j) = \max\{b_1(v_1), b_2(v_2), \cdots, b_n(v_n)\}$. Consider a direct selling mechanism $(p_i(v), c_i(v))$ such that $p_j(v) = 1$ for $v_j \geq b_j(v_j)$, $p_j(v) = 0$ for $v_j < b_j(v_j)$, and $p_i(v) = 0$ for $i \neq j$; and $c_i(v) = p_i(v)v_i - \int_0^{v_i} p_i(x, v_{-i})dx$ for all i. Obviously, $p_i(v)$ is non-decreasing in v_i, so is $\bar{p}_i(v_i)$, and $\bar{c}_i(v) = \bar{c}_i(0) + \bar{p}_i(v_i)v_i - \int_0^{v_i} \bar{p}_i(x)dx$ with $\bar{c}_i(0) = 0$. The mechanism is therefore incentive-compatible. By construction of the probability assignment functions, the object is assigned to the highest bidder. The winner pays $c_j(v) = p_j(v)v_j - \int_0^{v_j} p_j(x, v_{-j})dx = v_j - \int_{b_j(v_j)}^{v_j} 1dx = b_j(v_j)$, and the rest pay $c_i(v_i) = 0$. The outcome is exactly the same as the first-price auction.

In this mechanism, the distributions of the bidders' values are not assumed to be independent and symmetric either. However, the seller does have to know the exact distribution functions in order to construct $b_i(\cdot)$.

9.7. Being incentive compatible, it must be true that $u(v) = \max_{r\in[0,1]} u(r,v) = \bar{p}(r)v - \bar{c}(r)$. We show that $u(v,v)$ is convex. To see this, take $v_1, v_2 \in [0,1], v_1 \neq v_2$. For any $0 \leq \lambda \leq 1$, writing $\bar{v} = \lambda v_1 + (1\text{-}\lambda)v_2$, we have

$$\begin{aligned} u(\bar{v}) &= \max_r [\bar{p}(r)\bar{v} - \bar{c}(r)] \\ &= \max_r [\lambda(\bar{p}(r)v_1 - \bar{c}(r)) + (1\text{-}\lambda)(\bar{p}(r)v_2 - \bar{c}(r))] \\ &\leq \lambda \max_r [\bar{p}(r)v_1 - \bar{c}(r)] + (1\text{-}\lambda) \max_r [\bar{p}(r)v_2 - \bar{c}(r)] \\ &= \lambda u(v_1, v_1) + (1\text{-}\lambda)u(v_2, v_2). \end{aligned}$$

With the envelope theorem, $u'(v) = \bar{p}(v)$, and $u''(v) = \bar{p}'(v) \geq 0$ due to convexity of $u(\cdot)$.

9.8.

a). Let the equilibrium strategy of the all-pay with symmetric bidders be $b(v)$. First show that it is non-decreasing in v. A bidder with private value v_j wins the object with prob.$\{b(v_i) < b(v_j), \forall i \neq j\} \equiv G(b(v_j))$. If all the others employ the equilibrium strategy, then it is the best response for j to adopt the equilibrium strategy as well. Therefore we have $G(b(v_j))v_j - b(v_j) \geq G(b(z))v_j - b(z)$ and $G(b(z))z - b(z) \geq G(b(v_j))z - b(v_j)$. Adding the two inequalities together and rearranging, we obtain $(G(b(v_j)) - G(b(z)))(v_j - z) \geq 0$, hence $v_j > z$ if and only if $G(b(v_j)) \geq G(b(z))$. Furthermore, the distribution function is non-decreasing, thus $v_j > z$ if and only if $b(v_j) \geq b(z)$. We therefore conclude that $G = F^{N-1}$, and to bid $b(v)$ is equivalent to report v.

If a bidder with private value v bids z, his expected payoff is $F^{N-1}(z)v - b(z)$. The derivative of the payoff function w.r.t. z should be equal to zero when evaluated at $z = v$. Thus we have $(F^{N-1})'v = b'(v)$, which gives

$$b(v) = \int_0^v x dF^{N-1}(x).$$

Or, applying the revenue equivalence theorem, a bidder with private value v in the all-pay auction pays the same expected cost as in the first-price auction, which is $\int_0^v x dF^{N-1}(x)$. Since a bidder in the all-pay auction pays his own bid, winning or losing, his bid then must be $\int_0^v x dF^{N-1}(x)$.

b). In the first-price auction a bidder with v bids $\int_0^v x dF^{N-1}(x)/F^{N-1}(v)$, which is greater than $\int_0^v x dF^{N-1}(x)$, as $F^{N-1}(v) < 1$. Since a bidder in the all-pay auction has to pay his bid whether he wins or not, he bids lower than when he doesn't have to pay if he doesn't get the object.

c) and d). The seller's expected revenue is

$$N \int_0^1 \left[\int_0^v x dF^{N-1}(x) \right] f(v) dv = N(N-1) \int_0^1 v(1 - F(v)) F^{N-2}(v) f(v) dv,$$

which can be derived from the bids, as is shown above. Or it can be determined by applying the revenue equivalence principle.

9.9. Two bidders, second-price, all-pay auction.

Let $\beta(v)$ be the symmetric equilibrium bid. If the first bidder who has a private value v bids $\beta(z)$ or reports z while the second adopts the equilibrium strategy, she will win with probability $F(z)$ and lose with probability $1-F(z)$. In the former case she receives v and pays $\mathrm{E}[\beta(v_2) \mid v_2 < z]$, which is

$$\int_0^z \beta(x) \frac{f(x)}{F(z)} dx.$$

In the latter case, she pays her own bid $\beta(z)$. Her expected payoff is

$$F(z)\big(v - \int_0^z \beta(x) \frac{f(x)}{F(z)} dx\big) - (1 - F(z))\beta(z).$$

The derivative of the payoff w.r.t. z, evaluated at $z = v$, should be equal to zero. We have

$$v f(v) - \beta(v) f(v) + \beta(v) f(v) - (1 - F(v))\beta'(v) = 0,$$

which gives

$$\beta(v) = \int_0^v \frac{x f(x)}{1 - F(x)} dx.$$

Alternatively, applying the revenue equivalence theorem, we have

$$F(v) \int_0^v \beta(x) \frac{f(x)}{F(v)} dx + (1 - F(v))\beta(v) = \int_o^v x f(x) dx,$$

where the right-hand side of the above equation is the expected payment of a bidder with value v in a second price auction with 2 bidders. Differentiating the above identity w.r.t. v generates

$$(1 - F(v))\beta'(v) = vf(v),$$

which gives the same solution as we found earlier with a direct approach.

Still another alternative, which I had attempted, turned out to be incorrect but instructive. In this auction, the seller collects $\beta(\underline{v})$ twice, where $\underline{v} = \min\{v_1, v_2\}$, which has a density function $g(x) = 2(1 - F(x))f(x)$. Applying the revenue equivalence theorem, we have

$$\int_0^1 2\beta(v)g(v)dv = \int_0^1 vg(v)dv.$$

It is straightforward that

$$\beta(v) = \frac{v}{2},$$

which seems to make sense but is incorrect. The thrust of the theorem is equivalence of expected revenue from an individual with value v. Equivalence of the expected revenue over v follows as a result of weighted averaging. However, specifics can be lost in averaging.

9.10. The expected private value to a bidder with value v now is $v/2$. The symmetric equilibrium strategy should be such that $\frac{1}{2}F^{N-1}(z)v - \beta(z)$ is maximized at $z = v$. It is straightforward that the equilibrium bid is half of the regular first-price bid, which is $\int_0^v xdF^{N-1}(x)/(2F^{N-1}(v))$. The expected revenue is hence half of that in a first-price auction.

9.11. Let $r_0 > 0$ be the reserve. The probability assignment functions have to satisfy the following: $p_i(v) = 0$ for $v_i \leq r$, $i = 1, 2, ..., N$ and $p_0(v) = 1 - \sum p_i(v)$. Expression (9.12) on page 448 in the Text shall be modified as follows,

$$R = \int_0^1 \cdots \int_0^1 \left\{ \sum_{i=1}^N p_i(v) \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] + [1 - \sum_{i=1}^N p_i(v)]r_0 \right\} f(v)dv.$$

The revenue maximizing mechanism's assignment functions are as follows,

$$p_i^*(v) = \begin{cases} 1, & \text{if } v_i - \frac{1-F_i(v_i)}{f_i(v_i)} > \max\{r_0, v_j - \frac{1-F_j(v_j)}{f_j(v_j)}\} \text{ for all } j \neq i; \\ 0, & \text{otherwise.} \end{cases}$$

The reserve is determined by $r = r - \frac{1-F(r)}{f(r)}$ in the symmetric case. For the uniform distribution on $[0, 1]$, $r_0^* = 1/2$.

9.12. With the identical distributions, $MR_i(v_i) > MR_j(v_j)$ if and only if $v_i > v_j$, hence the assignment functions of the revenue-maximizing mechanism specified in 9.11 can be modified as

$$p_i^*(v) = \begin{cases} 1, & \text{if } v_i > \max\{r_0, v_j\} \text{ for all } j \neq i; \\ 0, & \text{otherwise,} \end{cases}$$

which is the same as those in the first-price auction with a reserve. Following the revenue equivalence theorem, the optimal reserve is the same as in 9.11.

9.13. Assume that the object is worth v_0, $0 \leq v_0 < 2$ to the seller. The assignment functions should be modified as

$$p_i^*(v) = \begin{cases} 1, & \text{if } v_i > \max\{v_0, \max\{r_0, v_j\}\} \text{ for all } j \neq i; \\ 0, & \text{otherwise,} \end{cases}$$

The optimal reserve is to be determined by

$$r_0 - \frac{1 - F(r_0)}{f(r_0)} = v_0.$$

For $F(v) = v - 1$ on $[1, 2]$, $(1 - F(v))/f(v) = 2 - v$. So $r_o^* = 1$ if $v_0 = 0$, $r_o^* = 1\frac{1}{2}$ if $v_0 = 1$.

9.14. $v_i \in [a_i, b_i]$ follows uniform distribution.

The virtual marginal revenue is as follows,

$$MR_i = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = 2v_i - b_i.$$

The optimal probability assignment rule that $p_i^* = 1$ if $MR_i > \max_{j \neq i}\{MR_j, 0\}$ and $p_i^* = 0$ otherwise translates to $p_i^*(v) = 1$ if

$$v_i > b_i/2 + \max_{j \neq i}\{0, v_j - b_j/2\},$$

and $p_i^*(v) = 0$ otherwise. But $v_i \geq a_i$, hence $p_i^*(v) = 1$ if $v_i > \max\{a_i, b_i/2 + \max_{j\neq i}\{0, v_j - b_j/2\}\}$. Let $c_i^*(a_i, v_{-i}) = 0$ for all v_{-i}, then

$$c_i^*(v) = p_i^*(v)v_i - \int_{a_i}^{v_i} p_i^*(x, v_{-i})dx_i = \max\{a_i, b_i/2 + \max_{j\neq i}\{0, v_j - b_j/2\}\}$$

for the winner, and $c_j^*(v) = 0$ for the others.

9.15.

a). In the second stage for player j to purchase if and only if $v_j > b_i$ is the best response to any b_i. In the first stage, given that i plays the specified strategy, a deviation such that $b_j' < b_j^*(v_j)$ changes the outcome only if $v_i < b_j^*(v_j)$ but $v_i > b_j'$. This change has no effect on j's payoff when $v_j < b_i^*$, but a loss when $v_j > b_i^*$. Similarly, a deviation that $b_j' > b_j^*(v_j)$ changes the outcome only if $v_i > b_j^*(v_j)$ but $v_i < b_j'$. This change has no effect on j's payoff when $v_j < b_i^*$. When $v_j > b_i^*$, the deviation would induce individual i to say no instead of yes, thus generate a gain. However, $v_j > b_i^*$ and $v_i > b_j^*$ can never occur, because $v_j > b_i^*(v_i)$ translates to $MR_j(v_j) > MR_j(b_i) = \max\{0, MR_i(v_i)\}$, while $v_i > b_j^*(v_j)$ implies $MR_i(v_i) > \max\{0, MR_j(v_j)\}$. The two inequalities $MR_i(v_i) > \max\{0, MR_j(v_j)\}$ and $MR_j(v_j) > \max\{0, MR_i(v_i)\}$ contradict each other.

b). The equilibrium maximizes the seller's revenues because $p^*(v)_i = 1$ if $MR_i(v_i) = \max\{MR_1(v_1), MR_2(v_2)\} > 0$, $p_i^*(v) = 0$ otherwise. It is not always efficient. It is not efficient, for example, if $v_0 = 0$, $v_i > 0$ but $MR_i(v_i) \leq 0$ for $i = 1, 2$, where v_0 is the seller's value of the object. It can also be inefficient if $v_i > b_j$ and $v_j < b_i$ but $v_i < v_j$.

c). It is essentially the equilibrium strategies of the second-price auction. It is always efficient.

d). It does not necessarily maximizes the seller's revenues because the object is assigned to the one whose value is equal to $\max\{v_1, v_2\}$, but not necessarily to the one who generates the greatest non-negative marginal revenue, if the distributions are not identical.

e). The strategies are the same as the original, except that MR_i and MR_j are replaced with g_i and g_j. If $g_i \equiv g_j$, it is efficient because $v_j > b_i$ if and only if $g(v_j) > g(v_i)$, which implies $v_j > v_i$. Here it is assumed that the value of the object to the seller is zero.

9.16. If F is convex, then $F'' = f' \geq 0$ (assuming differentiability). $(1 - F(x))/f(x))' = -1-(1-F(x))f'/f^2 < 0$, so $x-(1-F(x))/f(x)$ is increasing. Convexity of F is not necessary.

9.17. Let v_i follow distribution F_i on $[0, b_i]$, and v_i and v_j are mutually independent. The seller's value of the object is assumed to be $v_0 \geq 0$.

a). Efficient assignment functions must be such that $p_i(v) = 1$ for i with $v_i > \max\{v_0, v_j, \forall j \neq i\}$, and $p_i(v) = 0$ otherwise. It is non-decreasing in v_i as well.

b). The cost functions are as follows,

$$c_i(v) = c_i(0, v_{-i}) + p_i(v)v_i - \int_o^{v_i} p_i(x)dx, \text{ for } i = 1, 2, ..., N.$$

These cost functions satisfy

$$\bar{c}_i(v_i) = \bar{c}_i(0) + \bar{p}_i(v_i)v_i - \int_0^{v_i} \bar{p}_i(x)dx.$$

Therefore, $\{p_i(v), c_i(v)\}$ specified in (a) and (b) is an efficient, IR and IC direct selling mechanism.

c). Given the assignment functions, $c_i(v) = c_i(0, v_i) + \max\{v_0, v_j, \forall j \neq i\}$ for i such that $v_i > \max\{v_0, v_j, \forall j \neq i\}$, and $c_i(v) = c_i(0, v_{-i})$ otherwise. For individual rationality, $\bar{c}_i(0) \leq 0$. So the expected revenue is maximized at $\bar{c}_i(0) = 0$, which can be achieved by setting $c_i(0, v_{-i}) = 0$ for all i.

d). The efficient, IR and IC direct selling mechanism specified in (a)-(c) is exactly a second-price auction with a reserve equal to the seller's value. The English auction is strategically equivalent to a second-price auction, and hence exhibits the same properties. The first-price and Dutch auctions are not necessarily efficient if the the bidders are asymmetric.

9.18. Let $p_i(v), c_i(v)$ be a deterministic, IC direct selling mechanism, where private values are independent, taking values in $[0, b_i]$.

a). Suppose $p_j(v) = 1$ and $p_i(v) = 0$ for $i \neq j$ for a given v. Since p_i is non-decreasing, $p_i(x, v_{-i}) = 0$ for $x < v_i$ and $p_j(x, v_{-j}) = 0$ for some $x < v_j$, given v. Let $r_j(v_{-j}) \equiv \max\{x \in [0, b_j] \,|\, p_j(x, v_{-j}) = 0\}$. Now construct cost functions as follows,

$$c_i^*(v) = p_i(v)v_i - \int_0^{v_i} p_i(x, v_i)dx.$$

Clearly, $c_i^*(v) = 0$ for $i \neq j$, and $c_j^*(v) = r_j(v_{-j})$. That is, the winner pays $r_j(v_{-j})$, which is independent of v_j, and the others pay nothing.

Since the two mechanisms employ the same assignment functions, the expected revenues are the same, as long as $\bar{c}_i(0) = \bar{c}_i^*(0)$.

b) and c). Both first-price and all-pay, first-price auctions are deterministic. By the revelation principle, there exist incentive-compatible direct mechanisms that generate the same equilibrium outcomes as those auctions. These equivalent mechanisms therefore are deterministic as well. Then the result applies. In particular, if these auctions are symmetric, the required mechanism specified by this result is a second-price auction.

9.19. The probability assignment functions of the optimal direct selling mechanism are $p_i^*(v) = 1$ if $MR_i(v_i) > \max\{0, MR_j(v_j)\}$ for all $j \neq i$ and $p_i^*(v) = 0$ otherwise. The payment functions are

$$c_i^*(v) = p_i^*(v)v_i - \int_0^{v_i} p_i^*(x)dx = r^*(v_{-i}),$$

if $p_i^*(v) = 1$, and $c^*(v) = 0$ if $p_i^*(v) = 0$, where $r^*(v_{-i})$ is such that $MR_i(r^*) = \max_{j \neq i}\{0, MR_j(v_j)\}$. As MR_i is assumed to be strictly increasing, $v_i > r^*$.

If the winner submitted a report higher or lower than his true value but still got the object, his payoff would not be affected since his payment does not depend on his own report. If he submitted a sufficiently lower value, he would not be assigned the object and lose the net payoff $v_i - r^* > 0$. For the one who does not get the object in equilibrium, her payoff would not change if she instead submitted a report higher or lower than her true value but still didn't get the object. However, if she reported a sufficiently higher value and won the object, she would pay more than her value. To see this, suppose $p_i^*(v) = 1$ and $p_k^*(v) = 0$. So $MR_i(v_i) > MR_k(v_k)$ in equilibrium. If player k reported b_k so that $MR_i(v_i) < MR_k(b_k)$, then she got the object and had to pay $r^*(v_{-k})$, which is determined by $MR_k(r^*(v_{-k})) = MR_i(v_i)$. Since $MR_i(v_i) > MR_k(v_k)$ and MR_k is strictly increasing, $v_k < r^*(v_{-k})$.

9.20. Assume that $v_i \in [0, b_i]$ and v_i's are independent. Let ρ_i be such that $MR_i(\rho_i) = 0$. Such a $\rho_i \in (0, b_i)$ exists and is unique since $MR_i(0) < 0$, $MR_i(b_i) > 0$ and MR_i is strictly increasing. The seller keeps the object if and only if v is such that $MR_i(v_i) \leq 0$ for all i. The probability of that event is $\prod_{i=1}^{N} F_i(\rho_i)$, which is strictly positive since $\rho_i > 0$.

9.21. Under the assumption of symmetry, all the four auctions employ the same assignment functions that the object goes to the individual with the highest value. Again by symmetry, $v_i > v_j$ if and only if $MR(v_i) > MR(v_j)$. Therefore, the revenue-maximizing mechanism assigns the object to the individual with the highest value as well. Let r be such that $MR(r) = 0$. Then $MR(v_j) > \max_{i \neq j}\{0, MR(v_i)\}$ is equivalent to $v_j > \max_{i \neq j}\{r, v_i\}$. If the four auctions set r as the reserve, then they employ exactly the same assignment rule and hence are revenue maximizing.

9.22.

a). Since the equilibrium strategies are the same and increasing, the bidder with the highest value wins the auction.

b). Bidder i with value equal to zero wins only if $0 > v_j$ for all $j \neq i$, the probability of which is zero. Suppose $c_i(0, v_{-i}) > 0$, then bidder i can simply bid zero to reduce his cost, since the rule is such that no bidder ever pays more than his bid.

c). Now this auction employs the same assignment rule and $c_i(0, v_{-i}) = 0$ as the standard first-price or second-price auctions. Following the revenue equivalence theorem, the expected revenue is the same as those auctions, which is the expected value of the second largest value of v.

$$\begin{aligned}
R &= \int_0^1 vN(N-1)F^{N-2}(v)(1-F(v))f(v)dv \\
&= N(N-1)\int_0^1 v \cdot v^{2N-4}(1-v^2)2vdv \\
&= N(N-1)\frac{4}{4N^2-1} \\
&= 1 - \frac{4N-1}{4N^2-1}.
\end{aligned}$$

9.23.

a). Telling the true value is a weakly dominant strategy, following the same argument that is applied to a second-price auction.

b). The seller's expected revenue is twice of the expectation of the third-largest value of N random variables,

$$2E[Y^{[3|N]}] = 2\int_0^1 vg(v)dv = N(N-1)(N-2)\int_0^1 v(1-F(v))^2F(v)^{N-3}f(v)dv,$$

which is $2(N-2)/(N+1)$ if the distribution is uniform.
(Note that $g(v) = \frac{1}{2}\cdots$, not $\frac{1}{6}\cdots$.)

9.24.

a). With $F(v) = v, v \in [0,1]$ and $n = N/2$ bidders, the equilibrium strategy of the first-price auction is

$$\hat{b}(v) = \frac{1}{F^{n-1}(v)}\int_0^v xdF^{n-1}(x) = v - \int_0^v \frac{F^{n-1}(x)}{F^{n-1}(v)}dx = v - \frac{v}{n} = v - \frac{2v}{N}.$$

b). The expected revenue from each winner is the expected value of the second largest of n private values, which is

$$E[Y^{[2|n]} = \int_0^1 n(n-1)v(1-F(v))F^{n-2}(v)f(v)dv = \frac{n-1}{n+1} = \frac{N-2}{N+2}.$$

Clearly, $(N-2)/(N+1) > (N-2)/(N+2)$.

The result will be the same if a second-price auction is employed in the two rooms, by the revenue equivalence theorem.

Intuitive explanations? If both mechanisms employ the second-price auction, each bidder bids the same in both situations. The difference in revenues then is statistical in nature: The expectation of the third largest of $2N$ i.i.d. random variables is greater than that of the second largest of N i.i.d. random variables.

Suppose that the auction in 9.23 is a first-price one, that is, the top two bidders receive each object and pay their own bids, and the others pay nothing. Now the probability assignment functions are the same as those described in 9.23, and $c_i(0, v_{-i}) = 0$ in both auctions as well. Therefore, the expected revenue is to be the same as is found in 9.23. One might then argue that, with a larger number of participants, it is less likely to win the object. Therefore, the bidders bid more aggressively when there are more bidders. On the other hand, the bidders would bid less aggressively if the second-highest bid can also win the object. On balance, the statistical explanation still sounds more plausible.

9.25. Given that v_i follows uniform distribution on $[a_i, b_i]$,

$$MR_i(x) = x - \frac{1 - (x - a_i)/(b_i - a_i)}{1/(b_i - a_i)} = 2x - b_i.$$

Let the reserve price ρ_i be such that $MR_i(\rho_i) = 0$, or $\rho_i = b_i/2$.

The specified sealed bid is essentially a second-price auction, with a modification that the winner pays his reserve price in addition to the second-highest bid. Accordingly, the bidders will subtract $b_i/2$ from the bids that they would otherwise submit in a second price auction without this modification. It is known that truthful reporting is a weakly dominant strategy in a second price auction, the bid in the modified auction therefore is $\beta_i(v_i) = v_i - b_i/2$. Now $MR_i(v_i) = 2v_i - b_i = 2\beta(v_i)$, therefore, $p_i^*(v) = 1$ if $\beta_i > \max\{0, \beta_j\}$ for all $j \neq i$ is equivalent to $p_i^*(v) = 1$ if $MR_i(v_i) > \max\{0, MR_j(v_j)\}$ for all $j \neq i$, which maximizes the expected revenues.

9.26. Suppose $\hat{x}$ that maximizes $\sum v_i(x_i)$ were not Pareto efficient. Then there must exist a feasible allocation $y \in X$ and a set of transfers such that $v_i(y_i) + \tau_i \geq v_i(\hat{x}_i)$ with at least one strict inequality. Therefore

$$\sum v_i(y_i) + \tau_i = \sum v_i(y_i) > \sum v_i(\hat{x}_i),$$

where $\sum \tau_i = 0$ is simply the fact that the total transfers among the members themselves net zero. The above inequality contradicts that $\hat{x}$ maximizes $\sum v_i(x_i)$. Therefore $\hat{x}$ must be Pareto efficient.

9.28. It is understood that the object to be auctioned off is single and indivisible. Thus, $X = \{0, 1, 2, ..., N\}$ and $x \in X$ can be written as $i \in X$. Write $t_i = v_i$, $T_i = V_i$ and $T = V$, then the probability assignment and cost functions in Definition 9.4 become

$$p^i(v_0, v_i, ..., v_N) \text{ and } c_i(v_0, v_i, ..., v_N) \text{ for } i = 0, 1, ..., N,$$

with $\sum_{i=0}^{N} p^i(v) = 1$. It is exactly Definition 9.1.

In particular, the VCG mechanism with $X = \{0, 1, 2, ..., N\}$ and $v_i(x, t_i) = t_i p^x(t) = v_i p_i(v)$ is equivalent to the second-price auction.

9.29. The new ex post cost functions, with the superscript omitted, are as follows,

$$c'_i(t_i) = \bar{c}_i(t_i) - \frac{1}{N-1}\sum_{j\neq i} \bar{c}_j(t_j).$$

Obviously, $\sum_i c'_i(t_i) = 0$. The expected utility of type t_i who reports r_i in the new mechanism is

$$u_i^{VCG}(r_i, t_i) + \frac{1}{N-1}\sum_{i\neq j} \bar{c}_j,$$

while the expected utility in the mechanism in Theorem 9.11 is

$$u_i^{VCG}(r_i, t_i) + \bar{c}_{i+1}.$$

The two objective functions differ only by a constant. Therefore, Theorem 9.11 applies to the new mechanism as well. Moreover, as $\bar{c}_j \geq 0$, the new mechanism is individually rational as well.

9.30.

b). Given that individual 2 reports truthfully, when individual 1 reports his type truthfully, his gross utility is $v_1(\hat{x}(t), t_1 = 3) = t_1 + 5 = 8$ for $t_2 = 1, 2, 3, 4, 5, 6, 7$ and $v_1(\hat{x}(t), t_1 = 3) = 2t_1 = 6$ for $t_2 = 8, 9$. Therefore, the expected gross utility, $\bar{v}_1(3) = E_{t_2 \in T_2}[v_1(\hat{x}(t), 3)]$, is equal to $(8\times7+6\times2)/9 = 68/9$. If he instead reports $r_2 = 2$, $v_1(\hat{x}(r_1, t_2), t_1 = 3) = t_1 + 5 = 8$ for $t_2 = 1, 2, 3, 4, 5, 6, 7, 8$ and $v_1(\hat{x}(r_1, t_2), t_1 = 3) = 2t_1 = 6$ for $t_2 = 9$. Thus $\bar{v}_1(r_1 = 2, t_1 = 3) = (8 \times 8 + 6 \times 1)/9 = 70/9$. Values of $\bar{v}_1(r_1, t_1 = 3)$ for other values of reports can be similarly calculated and are presented in Table 1.

The last row of Table 1 displays $\bar{u}_1(r_1, t_1 = 3) = \bar{v}_1(r_1, t_1 = 3) - \bar{c}_1(r_1)$, where $\bar{c}_1(r_1)$ is the answer to (a). It reveals that reporting untruthfully can do no better. (Individual 1's net expected utility is $\bar{u}_1(r_1, t_1 = 3) + \bar{c}_2$, where $\bar{c}_2$ is a constant across r_1.)

Table 1: $v_1(r_1, 3, t_2)$, $\bar{v}_1(r_1, 3)$ and $\bar{u}_1(r_1, 3)$ for $r_1 = 1, 2, ...9$

r_1	1	2	3	4	5	6	7	8	9
t_2									
1	8	8	8	8	8	8	8	8	8
2	8	8	8	8	8	8	8	8	6
3	8	8	8	8	8	8	8	6	6
4	8	8	8	8	8	8	6	6	6
5	8	8	8	8	8	6	6	6	6
6	8	8	8	8	6	6	6	6	6
7	8	8	8	6	6	6	6	6	6
8	8	8	6	6	6	6	6	6	6
9	8	6	6	6	6	6	6	6	6
$\bar{v}_1(r_1, t_1 = 3)$	$\frac{72}{9}$	$\frac{70}{9}$	$\frac{68}{9}$	$\frac{66}{9}$	$\frac{64}{9}$	$\frac{62}{9}$	$\frac{60}{9}$	$\frac{58}{9}$	$\frac{56}{9}$
$\bar{c}_1(r_1)$	$\frac{10}{9}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	0	0	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{2}{3}$
$\bar{u}_1(r_1, t_1 = 3)$	$\frac{62}{9}$	$\frac{64}{9}$	$\frac{65}{9}$	$\frac{65}{9}$	$\frac{64}{9}$	$\frac{62}{9}$	$\frac{59}{9}$	$\frac{55}{9}$	$\frac{50}{9}$

However, truthful reporting is not a weakly dominant strategy, as is pointed out in the Text. Take a strategy profile $(r_1(\cdot), r_2(\cdot))$, where $r_2(t_2) = 1$ for all $t_2 = 1, 2, ..., 9$, and $r_1(3) = 3$. Now consider a deviation by player 1, r_1', where $r_1'(3) = 8$ and $r_1'(t_1) = r_1(t_1)$ for $t_1 \neq 3$. Individual 1 with $t_1 = 3$, reporting truthfully (playing $r_1(3)$), gets an expected payoff $\bar{v}_1(r_1(3), 3) = t_1 + 5 = 8$, as $\sum_i (r_i - 5) \leq 0$ and hence $\hat{x}(r_1(3), r_2(t_2)) = S$ for all t_2. Since neither individual is pivotal, $\bar{c}_1(r_1(3)) = 0$ and $c_2(r_1(3), r_2(t_2)) = 0$. If individual 1 plays $r_1'(3) = 8$ instead, the outcome still is $\hat{x}(r_1'(3), r_2(t_2)) = S$, and hence 1's payoff is the same $\bar{v}_1(r_1'(3), 3) = t_1 + 5 = 8$ for all t_2. His cost remains the same as well, $\bar{c}_1(r_1'(3)) = 0$, as he is not pivotal. But now individual 2 is pivotal and has to pay to individual 1 a cost equal to $c_2(r_1'(3), r_2(t_2)) = r_1'(3) - 5 = 3$. Thus, by deviating from $r_1(3)$ to $r_1'(3)$, individual 1 gets the same $\bar{v}_1$ and pays the same zero cost, but he receives 1/3 more from individual 2 of any t_2, as

$$\begin{aligned}
\bar{c}_2(r_1', r_2(t_2)) - \bar{c}_2(r_1, r_2(t_2)) &= \sum_{i=1}^{9} \frac{1}{9}[c_2(r_1'(i), r_2(t_2)) - c_2(r_1(i), r_2(t_2))] \\
&= \frac{1}{9}[c_2(r_1'(3), r_2(t_2)) - c_2(r_1(3), r_2(t_2))] \\
&= \frac{1}{3}.
\end{aligned}$$

c). Write $V(t) = \sum_{i=1}^{3}(t_i - 5)$ and $V_1(t_1) = t_2 + t_3 - 10$. If $V(t) > 0$ and $V_1(t_1) > 0$, or $V(t) \leq 0$ and $V_1(t_1) \leq 0$, then $c_1(t) = 0$; if $V(t) > 0$ and $V_1(t_1) \leq 0$, then $c_1(t) = -V_1(t_1)$; and if $V(t) \leq 0$ and $V_1(t_1) > 0$, then $c_1(t) = V_1(t_1)$. For all possible $t \in T_1 \times t_2 \times T_3$, $T_i = \{1, 2, ..., 9\}$, the costs to be paid by individual 1 are presented in Table 2, wherein the last row gives the expected costs, $\bar{c}_1(t_1) = E_{t_{-1} \in T_{-1}}[c_1(t)]$.

Table 2: $c_1(t)$ and $\bar{c}_1(t_1)$

	t_1	1	2	3	4	5	6	7	8	9
$t_2 + t_3$	Frequency									
2	1	0	0	0	0	0	0	0	0	0
3	2	0	0	0	0	0	0	0	0	0
4	3	0	0	0	0	0	0	0	0	0
5	4	0	0	0	0	0	0	0	0	0
6	5	0	0	0	0	0	0	0	0	0
7	6	0	0	0	0	0	0	0	0	3
8	7	0	0	0	0	0	0	0	2	2
9	8	0	0	0	0	0	0	1	1	1
10	9	0	0	0	0	0	0	0	0	0
11	8	1	1	1	1	0	0	0	0	0
12	7	2	2	2	0	0	0	0	0	0
13	6	3	3	3	0	0	0	0	0	0
14	5	4	0	0	0	0	0	0	0	0
15	4	0	0	0	0	0	0	0	0	0
16	3	0	0	0	0	0	0	0	0	0
17	2	0	0	0	0	0	0	0	0	0
18	1	0	0	0	0	0	0	0	0	0
	$\bar{c}_1(t_1)$	$\frac{20}{27}$	$\frac{40}{81}$	$\frac{22}{81}$	$\frac{8}{81}$	0	0	$\frac{8}{81}$	$\frac{22}{81}$	$\frac{40}{81}$

9.31.

a). If an individual has to contribute some time toward building either the pool or bridge, then he can enjoy the time that he doesn't have to give up, which is worth $k_i > 0$, if neither project is undertaken.

b). Assuming that individual i of type t_i reports r_1 while all the others report truthfully, his interim IR constraint is as follows

$$u_i(r_i, t_{-i}) = \sum_{t_{-i} \in T_{-i}} q_{-i}(t_{-i}) \sum_{x \in X} [\, p_i^x(r_i, t_{-i}) v_i(x, t_i) - c_i(r_i, t_{-i})] \geq k_i.$$

c). $\hat{x}(t) = S$ if $\sum_i (t_i + 5) \geq \max\{\sum_i 2t_i, \sum_i k_i\}$; $\hat{x}(t) = B$ if $\sum_i 2t_i > \max\{\sum_i (t_i + 5), \sum_i k_i\}$; and $\hat{x}(t) = D$ otherwise.

d). The sufficient condition for the existence of an IR and efficient mechanism for quasi-linear utility functions is

$$\sum_{t \in T} \sum_{i=1}^{N} q(t)(c_i^{VCG}(t) - \psi_i^*) = \sum_{i=1}^{N} (\bar{c}_i^{VCG} - \psi_i^*) \geq 0,$$

where

$$\psi_i^* = \max_{t_i \in T_i} [IR_i(t_i) - \sum_{t_{-i} \in T_{-i}} q_{-i}(t_{-i})(v_i(\hat{x}(t), t_i) - c_i^{VCG}(t))].$$

In the case where the individuals have no ownership rights over their leisure time, $\psi_i^* = 0$ for all i.

As $c_i^{VCG}(t) \geq 0$, the sufficient condition is always met for the case where the individuals have no ownership rights. The sufficient condition may be violated if there are ownership rights. Example 9.8 and Exercise 9.33(d) are two such examples.

The IR and efficient mechanism can be implemented by an IR-VCG mechanism, wherein each individual reports his type, the social state is $\hat{x}(t)$ and individual i pays

$$\bar{c}_i^{VCG}(t_i) - \psi_i^* - \bar{c}_{i+1}^{VCG}(t_{i+1}) + \bar{c}_{i+1}^{VCG} - \frac{1}{N} \sum_{j=1}^{N} (\bar{c}_j^{VCG} - \psi_j^*).$$

9.32. The assumption needed for this proposition is that, for every i, the value function $v_i(x, t_i)$ is non-decreasing in t_i for all states $x \in X$. Examples 9.3-9.7 satisfy this assumption.

With the VCG mechanism, given t, individual i's utility is

$$u_i(\hat{x}(t), t_i) = \sum_{\forall j} v_j(\hat{x}(t), t_j) - \sum_{j \neq i} v_j(\tilde{x}^i(t_{-i}), t_j).$$

Let t' be such that $t'_i \geq t_i$ and $t'_j = t_j$ for $j \neq i$. Clearly,

$$\Delta u_i = u_i(\hat{x}(t'), t'_i) - u_i(\hat{x}(t), t_i) = \sum_{\forall j} v_j(\hat{x}(t'), t'_j) - \sum_{\forall j} v_j(\hat{x}(t), t_j).$$

If $\hat{x}(t') = \hat{x}(t)$, then $\Delta u_i = v_i(\hat{x}(t'), t'_i) - v_i(\hat{x}(t), t_i) \geq 0$, since $v_i(x, t_i)$ is non-decreasing in t_i. If $\hat{x}(t') \neq \hat{x}(t)$, then

$$\sum_{\forall j} v_j(\hat{x}(t'), t'_j) \geq \sum_{\forall j} v_j(\hat{x}(t), t'_j) \geq \sum_{\forall j} v_j(\hat{x}(t), t_j),$$

where the first inequality is due to the fact that $\hat{x}(t')$ is efficient for t' while $\hat{x}(t)$ is not, and the second is the result previously derived for the case where $\hat{x}(t') = \hat{x}(t)$. Thus $u_i(\hat{x}(t), t_i)$ is non-decreasing in t_i for all $t_{-i} \in T_{-i}$, hence $E_{t_{-i} \in T_{-i}}[u_i(\hat{x}(t), t_i)]$ is non-decreasing as well.

9.33.

a). In Example 9.7, $IR_1(t_1) = 10$ for all t_1 and $IR_2(t_2) = 0$ for all t_2, which essentially introduces an additional state D such that $v_1(D, t_1) = 10$ and $v_2(D, t_2) = 0$ for all t_1 and t_2. Since $\hat{x}(t) > 10$ for all t, the allocation rule is not affected by replacing $IR_1(t_1) = 0$ with $IR_1(t_1) = 10$. Therefore, the VCG externality costs inflicted by individual 1 to individual 2 are not changed. However, the externality costs caused by the presence of 2 on 1 are different duo to $IR_1 = 10$. For example, when $t = (1, 1)$, $\hat{x}(t) = S$ and $v_1(S, 1) = 6$. But $\tilde{x}^1(1) = D$ and $v_1(D, 1) = 10$, hence $c_2^{VCG}(1, 1) = 10 - 6 = 4$. Table 3 lists $c_2^{VCG}(t)$ for all t and $\bar{c}_2^{VCG}(t_2)$.

Table 3: Values of $c_2^{VCG}(t_1, t_2)$ and $\bar{c}_2^{VCG}(t_2)$

t_1 \ t_2	1	2	3	4	5	6	7	8	9
1	4	4	4	4	4	4	4	4	4
2	3	3	3	3	3	3	3	3	6
3	2	2	2	2	2	2	2	4	4
4	1	1	1	1	1	1	2	2	2
5	0	0	0	0	0	0	0	0	0
6	1	1	1	1	0	0	0	0	0
7	2	2	2	0	0	0	0	0	0
8	3	3	0	0	0	0	0	0	0
9	4	0	0	0	0	0	0	0	0
$\bar{c}_2^{VCG}(t_2)$	$\frac{20}{9}$	$\frac{16}{9}$	$\frac{13}{9}$	$\frac{11}{9}$	$\frac{10}{9}$	$\frac{10}{9}$	$\frac{11}{9}$	$\frac{13}{9}$	$\frac{16}{9}$

b). $\bar{c}_1 = \sum \bar{c}_1^{VCG}(t_1)/9 = 10/27$, $\bar{c}_2 = 40/27$, and expected revenue $= 50/27$.

c). From the Text, $\psi_1^* = 46/9$ and $\psi_2^* = -34/9$. As $c_1 + c_2 > \psi_1^* + \psi_2^*$, the desired mechanism runs an expected surplus and can be implemented as follows. The allocation rule: $p^S(t) = 1$ if $t_1 + t_2 \leq 10$, and $p^B(t) = 1$ if $t_1 + t_2 > 10$. The costs of individual i of type t_i are

$$C_1(t_1) = \bar{c}_1(t_1) - 46/9 - \bar{c}_2(t_2) + 40/27 - (50/27 - 4/3)/2 = \bar{c}_1(t_1) - \bar{c}_2(t_2) - 35/9,$$

$$C_2(t_2) = \bar{c}_2(t_2) + 34/9 - \bar{c}_1(t_1) + 10/27 - (50/27 - 4/3)/2 = \bar{c}_2(t_2) - \bar{c}_1(t_1) + 17/27.$$

d). With IR_1=13, a social state, D=do not build either swimming pool or bridge, becomes relevant, and $\hat{x}(t)$ is as follows.

$$\hat{x}(t) = \begin{cases} D, & \text{if } t_1 + t_2 \leq 3; \\ S, & \text{if } 3 < t_1 + t_2 \leq 10; \\ B, & \text{otherwise.} \end{cases}$$

The VCG costs and expected costs are presented in Tables 4 and 5.

Table 4: Values of $c_1^{VCG}(t_1, t_2)$ and $\bar{c}_1^{VCG}(t_1)$

t_2 \ t_1	1	2	3	4	5	6	7	8	9
1	6	6	0	0	0	0	0	0	0
2	7	0	0	0	0	0	0	0	3
3	0	0	0	0	0	0	0	2	2
4	0	0	0	0	0	0	1	1	1
5	0	0	0	0	0	0	0	0	0
6	1	1	1	1	0	0	0	0	0
7	2	2	2	0	0	0	0	0	0
8	3	3	0	0	0	0	0	0	0
9	4	0	0	0	0	0	0	0	0
$\bar{c}_1^{VCG}(t_1)$	$\frac{23}{9}$	$\frac{12}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	0	0	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{6}{9}$

$$U_1^{VCG}(1) = E_{t_2 \in T_2}[v_1^{VCG}(1, t_2)] - \bar{c}_1^{VCG}(1) = 68/9 - 23/9 = 5,$$
$$U_2^{VCG}(1) = E_{t_1 \in T_1}[v_2^{VCG}(t_1, 1)] - \bar{c}_2^{VCG}(1) = 42/9 - 23/9 = 19/9.$$

$\psi_1^* = 13 - \min\{U_1^{VCG}(t_1) | t_1 \in T_1\} = 13 - U_1^{VCG}(1) = 8$, and $\psi_1^* = 0 - 19/9 = -19/9$. $\psi_1^* + \psi_2^* = 53/9$, while $\bar{c}_1^{VCG} + \bar{c}_2^{VCG} = 49/81 + 244/81 = 293/81 < 53/9$. The IR-VCG mechanism does not run a surplus.

Table 5: Values of $c_2^{VCG}(t_1, t_2)$ and $\bar{c}_2^{VCG}(t_2)$

t_1 \ t_2	1	2	3	4	5	6	7	8	9
1	0	0	7	7	7	7	7	7	7
2	0	6	6	6	6	6	6	6	9
3	5	5	5	5	5	5	5	7	7
4	4	4	4	4	4	4	5	5	5
5	3	3	3	3	3	3	3	3	3
6	2	2	2	2	1	1	1	1	1
7	2	2	2	0	0	0	0	0	0
8	3	3	0	0	0	0	0	0	0
9	4	0	0	0	0	0	0	0	0
$\bar{c}_2^{VCG}(t_2)$	$\frac{23}{9}$	$\frac{25}{9}$	$\frac{29}{9}$	$\frac{27}{9}$	$\frac{26}{9}$	$\frac{26}{9}$	$\frac{27}{9}$	$\frac{29}{9}$	$\frac{32}{9}$

9.34. The expected cost in the VCG mechanism is as follows.

$$\bar{c}_i^{VCG}(t_i) = \sum_{t_{-i} \in T_{-i}} q_{-i}(t_{-i}) c_i^{VCG}(t).$$

The expected costs of other related mechanisms are presented below.

$$\begin{aligned}
\bar{c}_i^{BB\text{-}VCG}(t_i) &= E[\bar{c}_i^{VCG}(t_i) - \bar{c}_{i+1}^{VCG}(t_{i+1})] \\
&= \bar{c}_i^{VCG}(t_i) - \sum_{t_{i+1} \in T_{i+1}} q_{i+1}(t_{i+1}) \bar{c}_{i+1}^{VCG}(t_{i+1}) \\
&= \bar{c}_i^{VCG}(t_i) - \bar{c}_{i+1}^{VCG}, \\
\bar{c}_i^{IR\text{-}VCG}(t_i) &= E[\bar{c}_i^{VCG}(t_i) - \psi_i^* - \bar{c}_{i+1}^{VCG}(t_{i+1}) + \bar{c}_{i+1}^{VCG} - \frac{1}{N} \sum_{j=1}^{N} (\bar{c}_j^{VCG} - \psi_j^*)] \\
&= \bar{c}_i^{VCG}(t_i) - \psi_i^* - \frac{1}{N} \sum_{j=1}^{N} (\bar{c}_j^{VCG} - \psi_j^*).
\end{aligned}$$

The cost function $c_i^B(t_i)$ introduced in the Text on page 474 is simply $c_i^{IR\text{-}VCG}(t_i)$ with all $\psi_i^* = 0$. Clearly, all $\bar{c}_i^{BB\text{-}VCG}(t_i)$, $\bar{c}_i^B(t_i)$ and $\bar{c}_i^{IR\text{-}VCG}(t_i)$ are $\bar{c}_i^{VCG}(t_i)$ plus a distinct constant.

9.35. Let $(p_{Ai}^x(t), c_{Ai}(t))$ and $(p_{Bi}^x(t), c_{Bi}(t))$ be two mechanisms such that $\bar{p}_{Ai}^x(t_i) = \bar{p}_{Bi}^x(t_i)$ for all $x \in X$ and all i. If $u_i(0|A) = u_i(0|B)$, then $\bar{c}_{Ai}(0) =$

$\bar{c}_{Bi}(0)$, since $u_i(0|k) = \sum_{x\in X} \bar{p}^x_{ki}(0)v_i(x,0) - \bar{c}_{ki}(0)$, $k = A, B$, and $\bar{p}^x_{Ai}(0) = \bar{p}^x_{Bi}(0)$. That $\bar{c}_{Ai}(t_i) = \bar{c}_{Bi}(t_i)$ follows immediately from Theorem 9.14, which states that $\bar{c}_{Ai}(t_i) - \bar{c}_{Ai}(0) = \bar{c}_{Bi}(t_i) - \bar{c}_{Bi}(0)$.

It follows directly that

$$E[R_A] = \sum_{i=1}^{N} E[\bar{c}_{Ai}(t_i)] = \sum_{i=1}^{N} E[\bar{c}_{Bi}(t_i)] = E[R_B].$$

9.36.

a). Ignoring the zero probability event of $t_1 = t_2$, let the probability assignment function be $p^I_1(t) = 1$ if $t_1 > t_2$ and $p^I_1(t) = 0$ otherwise, where state I refers to sole ownership by individual 1. Thus the VCG externality costs are $c^{VCG}_i(t) = t_j$ if $t_i > t_j$, and zero otherwise, $i, j = 1, 2, i \neq j$.

$$\bar{c}^{VCG}_i(t_i) = \int_o^{t_i} t_j dt_j = \frac{1}{2}t_i^2, \text{ and } U^{VCG}_i(t_i) = \int_0^{t_i} t_i dt_j - \bar{c}^{VCG}_i(t_i) = \frac{1}{2}t_i^2.$$

$$\psi^*_i(\alpha_i) = \max_{t_i\in[0,1]} (\alpha_i t_i - \frac{1}{2}t_i^2) = \frac{1}{2}\alpha_i^2.$$

It is known that $E[\bar{c}^{VCG}_i(t_1)]+E[\bar{c}^{VCG}_2(t_2)] = 1/3$. For the specified mechanism, it is required that $\alpha_1^2/2 + (1-\alpha_1)^2/2 \leq 1/3$, or

$$\frac{1}{2} - \frac{\sqrt{3}}{6} \leq \alpha_1 \leq \frac{1}{2} + \frac{\sqrt{3}}{6}.$$

With asymmetric ownership shares, it requires greater compensations to induce the individuals with larger shares to give up the status quo, and hence is harder to implement an IR and efficient allocation.

b). For N partners with ownership shares α_i for partner i, the efficient probability assignment functions are $p^j_j(t) = 1$ if $t_j > t_i$ for all $i \neq j$, and $p^j_i(t) = 0$ otherwise; and $p^k_i(t) = 0$ for $k \neq j$ and all i.

The efficient probability assignment rule allocates the business to the partner with largest t_i of t. Let t^I and t^{II} denote the largest and the second largest value, respectively, of $t_1, t_2, ..., t_N$, then $c_i^{VCG}(t) = t^{II}$ if $t_i = t^I$, $c_i^{VCG}(t) = 0$ otherwise.

$$\bar{c}_i^{VCG}(t_i) = \int_0^{t_i} x(N-1)F^{N-2}(x)f(x)dx = \frac{N-1}{N}t_i^N.$$

$$E[R] = \sum_i \int_0^1 \bar{c}_i^{VCG}(t_i)dt_i = \frac{N-1}{N+1}.$$

$$U_i^{VCG}(t_i) = \int_0^{t_i} t_i(N-1)F^{N-2}(x)f(x)dx - \bar{c}_i^{VCG}(t_i) = \frac{1}{N}t_i^N.$$

$$\psi_i^*(\alpha_i) = \max_{t_i \in [0,1]} (\alpha_i t_i - \frac{1}{N}t_i^N) = \frac{N-1}{N}\alpha_i^{\frac{N}{N-1}}.$$

As $(\psi_i^*)' = \alpha_i^{\frac{1}{N-1}}$ and $(\psi_i^*)'' = \frac{1}{N-1}\alpha_i^{\frac{2-N}{N-1}} > 0$, $\psi_i^*(\alpha_i)$ is convex in α_i, so is $\Psi(\alpha) = \sum_i \psi_i^*(\alpha_i)$. Given α and α', $\alpha \neq \alpha'$, let $\alpha^\lambda = \lambda\alpha + (1-\lambda)\alpha'$, $0 < \lambda < 1$, then $\max\{\alpha_i^\lambda\} < \max\{\max\{\alpha_i\}, \max\{\alpha_i'\}\}$ and $\min\{\alpha_i^\lambda\} > \min\{\min\{\alpha_i\}, \min\{\alpha_i'\}\}$, reflecting reduced asymmetry of ownership shares. And convexity of Ψ implies that $\Psi(\alpha^\lambda) \leq \lambda\Psi(\alpha) + (1-\lambda)\Psi(\alpha') \leq \max\{\Psi(\alpha), \Psi(\alpha')\}$. In particular, for a perfectly symmetric distribution of shares, i.e., $\bar{\alpha}$ such that $\bar{\alpha}_i = 1/N$ for all i, $\Psi(\bar{\alpha}) \leq \Psi(\alpha)$ for all α. In conclusion, greater symmetry in ownership shares reduces subsidies and makes it easier to implement an efficient allocation.

We have yet to show that there exists a non-empty set A in the $(N-1)$-dimensional simplex, such that $\Psi(\alpha) \leq E[R]$ for $\alpha \in A$. Define $\rho(N)$ as follows,

$$\rho(N) \equiv \ln\frac{\Psi(\bar{\alpha})}{E[R]} = \ln\frac{(\frac{1}{N})^{\frac{N}{N-1}}}{\frac{1}{N+1}} = \ln(N+1) - \ln N - \frac{1}{N-1}\ln N.$$

Expanding $\ln(N+1)$ around N, we obtain

$$\ln(N+1) = \ln N + \frac{1}{N} - \frac{1}{2N^2}\delta^2(N),$$

where $0 < \delta(N) < 1$. Thus,

$$\rho(N) = \frac{1}{N} - \frac{1}{2N^2}\delta^2(N) - \frac{1}{N-1}\ln N.$$

As $\ln N > 1$ for $N \geq 3$, it is clear that $\rho(N) < 0$, or $\Psi(\bar{\alpha}) < E[R]$, for $N \geq 3$. We've already found in (a) that $\Psi(\bar{\alpha}) < E[R]$ for $N = 2$. Therefore, for $N \geq 2$, such a set A exists around $\bar{\alpha}$. That is, balanced budget can be implemented for ownership shares $\alpha \in A$.