Appendix: Solving $f_\infty(z) = \int_a^b g_\tau(z|x) f_\infty(x) dx$

Assume that the solution exists and partition $[a, b]$ into n non-overlapping intervals $[s_{i-1}, s_i]$, $i = 1, 2, \ldots n$, such that $s_i = s_{i-1} + \frac{b-a}{n}$ with $s_0 = a$. Define z_j to be the midpoint of $[s_{j-1}, s_j]$. For any x, $g_\tau(z|x)$ is a probability density function implying $\int_a^b g_\tau(z|x) dz = 1$ so that we can write

$$\sum_{j=1}^{n} g_\tau(z_j|x) \frac{b-a}{n} \approx 1 \tag{1}$$

for any $x \in [a, b]$ where the approximation can be made arbitrarily accurate by taking n sufficiently large. Take $x = x_i$, the midpoint of $[s_{i-1}, s_i]$ and define $p_{ij} = g_\tau(z_j|x_i) \frac{b-a}{n} \geq 0$ for $j = 1, 2, \ldots n$. By virtue of (1) and the nonnegativity of the p_{ij}, we can, for any i, treat $\{p_{ij}\}_{j=1}^{n}$ as a (conditional) probability mass function. Define the matrix P by

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

and note that P has the same structure as the transition matrix of a Markov chain. We can use an argument similar to that used to motivate (1) to write $f_\infty(z) = \int_a^b g_\tau(z|x) f_\infty(x) dx$ as

$$f_\infty(z_j) \approx \sum_{i=1}^{n} g_\tau(z_j|x_i) f_\infty(x_i) \frac{b-a}{n} \tag{2}$$

and also to write

$$\sum_{j=1}^{n} f_\infty(z_j) \frac{b-a}{n} \approx 1.$$

Define $\phi_i = \frac{b-a}{n} f(x_i) = \frac{b-a}{n} f(z_i)$ for $i = 1, 2, \ldots n$ and write (2) as

$$\phi_j = \sum_{i=1}^{n} p_{ij} \phi_i. \tag{3}$$

By defining $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^{'}$, (3) is recognized as the expression for the product of $\phi^{'}$ and the i^{th} column of P so that we have $\phi^{'} = \phi^{'} P$. As P has the same structure as the transition matrix of a Markov chain, we recognize ϕ to be the ergodic mass function associated with that chain. Given P, it is straightforward to find ϕ (if it exists) and then use $f(x_i) = \phi_i / \frac{b-a}{n}$ $i = 1, 2, \ldots n$ to get a vector of values the ergodic density, $f_\infty(x)$, evaluated at a set of points $\{x_i\}_{i=1}^{n}$.