Notes: Review of Future & Present Value, Some Statistics & Calculating Security Returns

I. Future Values

How much is money today worth in the future? This is the future value (FV) of money today.

a) Simple Case: Interest compounded annually

Suppose the interest rate is 5% per year. How much is \$100 today worth one year from now?

FV= \$100 + \$100*(0.05) → combining terms

FV= \$100* (1.05) = \$105

What about two years from today?

FV= \$100* (1.05)(1.05) → simplifying

FV= $\$100 * (1.05)^2$ = \$110.25

What about N years from today?

FV= $\$100 * (1.05)^N$

b) More complex case: Interest compounded sub-annually

Suppose interest is compounded more than once a year. How to we calculate future values in this case? Let's first define some terms

Let r = annual interest rate

N= # of years

x= amount today

m = the # of times a year the interest rate is compounded

(ie: monthly compounding m=12

quarterly compounding m=4)

The general formula is:

FV= $\$x * (1+\frac{r}{m})^{mN}$ → that is, how much is $\$x$ worth N years in the future, with an annual rate of r compounded at a frequency of m.

case I: Quarterly Compounding

m=4, N=1, x=\$100, r=5%

FV= $\$100 * (1+\frac{0.05}{4})^{4}$ = \$105.0945

What about the value in 2 years?

FV= $\$100 * (1+\frac{0.05}{4})^{4*2}$ = \$110.449

case II: Monthly Compounding

m=12, N=1, x=\$100, r=5%

FV= $\$100 * (1+\frac{0.05}{12})^{12}$ = \$105.116

What about the value in 2 years?

FV= $\$100*(1+\frac{0.05}{12})^{12*2}$ = \$110.494

<u>caseIII: Continuous Compounding</u>

We can continue this trend, and allow m, the frequency of compounding, to grow larger and larger. The extreme case is when m goes to infinity; that is, when interest is continuously compounded. The future value in this case is:

FV= $\lim_{m\to\infty} \$x*(1+\frac{r}{m})^{mN}$ = $\$x*e^{rN}$

ex: N=1, x=\$100, r=5%

FV= $\$100*e^{0.05*1}$ = \$105.127

N=2, x=\$100, r=5%

FV= $\$100*e^{0.05*2}$ = \$110.517

II. Present Values

The difficulty with future values is that decisions about investments, and actual or potential future payments, are made today. Therefore, it would be helpful to know the value of those future payments today. That is, we want to know how much money in the future is worth today; the present value (PV) of future payments.

Example: I offer you the choice between \$100 today and \$100 tomorrow (where tomorrow is one year from today). Which one do you choose? ⇒ \$100 TODAY

Why? It's worth more because you can invest the \$100 today and earn interest on it for a year. As a result, you will end up with more than \$100 tomorrow. For example, if interest rates are 5%, \$100 today is worth \$105 tomorrow.

So now let's change the question. What is \$100 tomorrow worth today? What is the present value of \$100 received a year from today (when interest rates are 5%)?

We know that \$100 a year from today equals the present value times 1.05

⇒ \$100= PV*(1.05) rearrange to solve for PV

⇒ PV = $\frac{\$100}{1.05}$ = \$95.238

We can check that this is the correct present value. This amount, invested for a year at the prevailing interest rate, should equal \$100. ⇒ \$95.238*(1.05)= \$100

Now, let's expand this to ask what \$100 two years from now is worth today. That is, what is the present value of \$100 received two years from today? The interest rate=5%

PV= $\frac{\$100}{(1.05)^2}$ = \$90.7029

We can generalize this process to ask the present value of \$100 N years from today.

PV= $\frac{\$100}{(1.05)^N}$

Note: by answering how much future payments are worth today you are also answering how much you would pay today to receive payments in the future. How much would you pay to receive \$100 two years from today? You would be willing to pay what this payment is worth, and that is the present value.

Example: Perpetuities

A perpetuity is an asset that promises to pay a fixed amount during each period (usually a year) into infinity. How much would you pay for this asset?

Let x = annual payment

r = annual interest rate

You would be willing to pay the present value of the stream of future payments.

$$\Rightarrow \text{ValueofAnnuity}= \$x+\frac{\$x}{1+r}+\frac{\$x}{(1+r)^2}+\frac{\$x}{(1+r)^3}+\frac{\$x}{(1+r)^4}+....$$

$$= \$x*\left[1+\frac{1}{1+r}+\frac{1}{(1+r)^2}+\frac{1}{(1+r)^3}+\frac{1}{(1+r)^4}+....\right]$$

We can summarize this expression as follows:

$$= \$x*\sum_{z=0}^{\infty}\frac{1}{(1+r)^z}$$

Now, we have a formula for an infinite sum that allows us to simplify this Expression

$$= \$x*\frac{1}{1-\frac{1}{1+r}}=\$x*\frac{1}{\frac{1+r-1}{1+r}}$$

$$= \$x*\frac{1+r}{r}$$

III. Random Variables and Probability Distributions

A random variable is when a numerical value can be assigned to each exclusive outcome.

-in the context of investment decision making the payoff to the investment decision is a random variable. It can be denominated in dollars or as a percentage return.

Let X denote a random variable. An individual realization of the random variable can be denoted by x. The probability that a given value of x is realized is denoted by Pr(x).

The set, or list, of all possible values of a random variable with their associated probabilities is called the probability distribution of the random variable.

Note: if a value cannot be realized, its probability = 0 (Pr(x)=0). Also, all probabilities must sum to 1.

Example 1: Expected Return on a Stock

State of the World	Probability State is Realized	Return in Given State
Slump	0.2	20%
Normal	0.5	30%
Boom	0.3	50%

Example 2: Expected Year End Stock Price for Company Z

State of the World	Probability State is Realized	Price in Given State
Slump	1/3 = 0.33333	80

Normal	1/3 = 0.33333	110
Boom	1/3 = 0.33333	140

Expected Value

The logical question is what is the expected outcome of a draw from the probability distribution? What is the mean?

Ex1: expected return=mean return
= Pr(slump)*$\text{Return}_{\text{slump}}$ + Pr(normal)*$\text{Return}_{\text{normal}}$ + Pr(boom)*$\text{Return}_{\text{boom}}$
= 0.2*20%+0.5*30%+0.3*50%
= 34%

General Formula: E(r)= $\sum_{i=1}^{N} \Pr(state_i) * \text{Re}\,turn_{statei}$ where N is the # of total outcomes

Ex2: expected year-end stock price of Company Z
= 0.333*$80 + 0.333*$110 + 0.333*$140 = $110

Variance

We are also interested in the dispersion of a distribution.
One measure of dispersion is the range, the highest possible value in the distribution minus the lowest possible value. The difficulty with the range is that is dominated by outliers (that is, one very high or very low value can lead to a large range, although the remaining values in the distribution can be closely spaced together).

A more reasonable measure would be something that calculates that expected deviation of the r.v. from its expected value. However, this is, by definition, zero. However, the expected squared deviation from the expected value is not zero. This is called the *variance* and is defined as

$$Var(X) = \sigma^2(x) = E[X - E(X)]^2 = \sum_{i=1}^{N} \Pr(x_i)(x_i - E(X))^2$$

Let's consider two probability distributions for the year-end stock price for Company Z. The first is the distribution given in example 2 above. The second is:

State of the World	Probability State is Realized	Price in Given State
Slump	1/3	60
Normal	1/3	100
Boom	1/3	170

It is easy to find that the expected year-end price is $110, similar to the first distribution. Let's compare the variance of both distributions.

Variance of 1st distribution= $(1/3)(80-110)^2 + (1/3)(110-110)^2 + (1/3)*(140-110)^2$
= 600

Variance of 2nd distribution= $(1/3)(60-110)^2 + (1/3)(100-110)^2 + (1/3)*(170-110)^2$
= 2066.67

By squaring the deviations, however, the units of the variance are also squared. In the example, the units of the variance are prices squared. One way to deal with this is to instead focus on the square root of the variance, which is called the standard deviation, and is denoted by σ.

$$\sigma(X) = \sqrt{\sigma^2(X)}$$

Standard deviation of 1st distribution = $\sqrt{600}$ = 24.5
Standard deviation of 2nd distribution = $\sqrt{2066.67}$ = 45.46

IV. Calculating Security Returns

A. Holding Period Return
One of the most basic calculations for securities is the realized rate of return. The most general calculation, and one the book refers to frequently, is the holding period return, which equals the return received from a security over the time it is held. For example, if an investor holds a security for 6 months, the holding period return is the 6 month return. If an investor holds a security for 2 years, the holding period return is the 2 year return.
The holding period return is calculated as follows:

$$HoldingPeriod\ \mathrm{Re}turn = \frac{End\ \mathrm{Pr}ice_{security} - Beginning\ \mathrm{Pr}ice_{security} + AnyCashFlow}{Beginning\ \mathrm{Pr}ice_{security}}$$

In words, the rate of return is what you sell the security for minus what you bought it for plus any cash flow, such as dividends or coupon payments, all as a percentage of what you paid for the security. You can also think of it as the amount you earn from the security – amount you paid for the security over your initial investment.

$$RateOf\ \mathrm{Re}turn = \frac{AmountEarned - AmountPaid\,(InitialInvestment)}{InitialInvestment} = \frac{\mathrm{Pr}ofit}{InitialInvestment}$$

B. Annual Rate of Return
Investors usually care about one particular holding period return, which is the annual rate of return. This is the rate of return an investor will make by holding the security for a year, and is the standard measure of the return on a security. It is calculated in the exact same way as the holding period return and is the most common return cited for securities.

C. Rate of Return vs. Return
You will notice in this class that I will become sloppy about using the terminology *rate of return* and will oftentimes simply refer to the *return* on an investment. This is done on purpose, as investors usually refer to rates of returns on investments as simply the return on an investment. This occurs although, technically, the return is the profit on the investment. Thus **it is important for you to realize that return is broadly used to mean the rate of return on an investment and NOT the profit.**

Why do we care about rates of return rather than simply the profit, or return, from an investment? Why is return used so sloppily? The following example will illustrate why.

Consider two investments:

1. The first investment involves buying 100 shares of a stock that has a price of $20/share.

Beg. Price per share	$20
# of shares	100
Initial Investment	$2,000

Now let's say that at the end of a year the price per share goes up to $30, and the investor sells.

End Price per share	$30
# of shares	100
End Value	$3,000

The profit on this investment = $1,000

The return, however, = $\frac{\$3{,}000 - \$2{,}000}{\$2{,}000}$ = 50%

2. The second investment involves buying 10,000 shares of another stock that has a price of $20/share

Beg. Price per share	$20
# of shares	10,000
Initial Investment	$200,000

Now let's say that at the end of a year the price per share goes up to $20.1, and the investor sells.

End Price per share	$20.1
# of shares	10,000
End Value	$201,000

The profit on this investment = $1,000

The return, however, = $\frac{\$201{,}000 - \$200{,}000}{\$200{,}000}$ = 1%

Notice that on the basis of profits alone, these two investments are equal, implying that investors would be indifferent between the two. However, by looking at the returns we can see quite clearly that Investment 1 is dramatically better than Investment 2. With Investment 1 the investor only had to put in $2,000 to make $1,000- yielding a return of 50%. However, with Investment 2 the investor had to put in $200,000 to make $1,000- yielding a return of only 1%. Clearly any rational investor would prefer Investment 1.