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THE STRUCTURE OF SIMPLE GENERAL EQUILIBRIUM MODELS[1]

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I. INTRODUCTION

It is difficult to find any major branch of applied economics that has not made some use of the simple general equilibrium model of production. For years this model has served as the work-horse for most of the developments in the pure theory of international trade. It has been used to study the effects of taxation on the distribution of income and the impact of technological change on the composition of outputs and the structure of prices. Perhaps the most prominent of its recent uses is to be found in the neo-classical theory of economic growth.

Such intensive use of the simple two-sector model of production suggests that a few properties are being retranslated in such diverse areas as public finance, international trade, and economic growth. The unity provided by a common theoretical structure is further emphasized by the dual relationship that exists between sets of variables in the model itself. Traditional formulations of the model tend to obscure this feature. My purpose in this article is to analyze the structure of the simple competitive model of production in a manner designed to highlight both the dual relationship and the similarity that exists among a number of traditional problems in comparative statics and economic growth.

The model is described in Sections II and III. In Section IV I discuss the dual nature of two theorems in the theory of international trade associated with the names of Stolper and Samuelson on the one hand and Rybczynski on the other. A simple demand relationship is added in Section V, and a problem in public finance is analyzed—the effect of excise subsidies or taxes on relative commodity and factor prices. The static model of production is then reinterpreted as a neo-classical model of economic growth by letting one of the outputs serve as the capital good. The dual of the "incidence" problem in public finance in the static

[1] I am indebted to the National Science Foundation for support of this research in 1962–64. I have benefited from discussions with Hugh Rose, Robert Fogel, Rudolph Penner, and Emmanuel Drandakis. My greatest debt is to Akihiro Amano, whose dissertation, *Neo-Classical Models of International Trade and Economic Growth* (Rochester, N.Y.: University of Rochester, 1963), was a stimulus to my own work.

model is shown to have direct relevance to the problem of the stability of the balanced growth path in the neoclassical growth model. In the concluding section of the paper I show how these results can be applied to the analysis of technological progress. Any improvement in technology or in the quality of factors of production can be simply viewed as a composite of two effects, which I shall term the "differential industry" effect and the "differential factor" effect. Each effect has its counterpart in the dual problems discussed in the earlier part of the paper.

II. THE MODEL

Assume a perfectly competitive economy in which firms (indefinite in number) maximize profits, which are driven to the zero level in equilibrium. Consistent with this, technology in each of two sectors exhibits constant returns to scale. Two primary factors, labor (L) and land (T), are used in producing two distinct commodities, manufactured goods (M) and food (F). Wages (w) and rents (r) denote the returns earned by the factors for use of services, whereas p_M and p_F denote the competitive market prices of the two commodities.

If technology is given and factor endowments and commodity prices are treated as parameters, the model serves to determine eight unknowns: the level of commodity outputs (two), the factor allocations to each industry (four), and factor prices (two). The equations of the model could be given by the production functions (two), the requirement that each factor receive the value of its marginal product (four), and that each factor be fully employed (two). This is the format most frequently used in the theory of international trade and the neoclassical theory of growth.[2] I consider, instead, the formulation of the model suggested by activity analysis.

The technology is described by the columns of the A matrix,

$$A = \begin{pmatrix} a_{LM} & a_{LF} \\ a_{TM} & a_{TF} \end{pmatrix},$$

where a_{ij} denotes the quantity of factor i required to produce a unit of commodity j. With constant returns to scale total factor demands are given by the product of the a's and the levels of output. The requirement that both factors be fully employed is thus given by equations (1) and (2). Similarly, unit costs of production in each industry are given by the columns of A multiplied by the factor prices. In a competitive equilibrium with both goods being produced, these unit costs must reflect market prices, as in equations (3) and (4).[3] This formulation serves to emphasize the dual relationship between factor endowments and commodity outputs on the one hand

$$a_{LM}M + a_{LF}F = L\,, \qquad (1)$$

$$a_{TM}M + a_{TF}F = T\,, \qquad (2)$$

$$a_{LM}w + a_{TM}r = p_M\,, \qquad (3)$$

$$a_{LF}w + a_{TF}r = p_F\,, \qquad (4)$$

[2] As an example in each field see Murray C. Kemp, *The Pure Theory of International Trade* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1964), pp. 10–11; and J. E. Meade, *A Neo-Classical Theory of Economic Growth* (London: Allen & Unwin, 1961), pp. 84–86.

[3] These basic relationships are usually presented as inequalities to allow for the existence of resource(s) in excess supply even at a zero price or for the possibility that losses would be incurred in certain industries if production were positive. I assume throughout that resources are fully employed, and production at zero profits with positive factor and commodity prices is possible. For a discussion of the inequalities, see, for example, R. Dorfman, Paul A. Samuelson, and Robert M. Solow, *Linear Programming and Economic Analysis* (New York: McGraw-Hill Book Co., 1958), chap. xiii; or J. R. Hicks, "Linear Theory," *Economic Journal*, December, 1960.

(equations [1] and [2]) and commodity prices and factor prices on the other (equations [3] and [4]).

In the general case of variable coefficients the relationships shown in equations (1)–(4) must be supplemented by four additional relationships determining the input coefficients. These are provided by the requirement that in a competitive equilibrium each a_{ij} depends solely upon the ratio of factor prices.

III. THE EQUATIONS OF CHANGE

The comparative statics properties of the model described in Section II are developed by considering the effect of a change in the parameters on the unknowns of the problem. With unchanged technology the parameters are the factor endowments (L and T) and the commodity prices (p_M and p_F), the right-hand side of equations (1)–(4).

Let an asterisk indicate the relative change in a variable or parameter. Thus p_F^* denotes dp_F/p_F and L^* denotes dL/L.[4] The four equations in the rates of change are shown in (1.1) through (4.1):

$$\lambda_{LM}M^* + \lambda_{LF}F^* = L^* - [\lambda_{LM}a_{LM}^* + \lambda_{LF}a_{LF}^*], \quad (1.1)$$

$$\lambda_{TM}M^* + \lambda_{TF}F^* = T^* - [\lambda_{TM}a_{TM}^* + \lambda_{TF}a_{TF}^*], \quad (2.1)$$

$$\theta_{LM}w^* + \theta_{TM}r^* = p_M^* - [\theta_{LM}a_{LM}^* + \theta_{TM}a_{TM}^*], \quad (3.1)$$

$$\theta_{LF}w^* + \theta_{TF}r^* = p_F^* - [\theta_{LF}a_{LF}^* + \theta_{TF}a_{TF}^*]. \quad (4.1)$$

The λ's and θ's are the transforms of the a's that appear when relative changes are shown. A fraction of the labor force is used in manufacturing (λ_{LM}), and this plus the fraction of the labor force used in food production (λ_{LF}) must add to unity by the full-employment assumption (shown by equation [1]). Similarly for λ_{TM} and λ_{TF}. The θ's, by contrast, refer to the factor shares in each industry. Thus θ_{LM}, labor's share in manufacturing, is given by $a_{LM}\, w/p_M$. By the zero profit conditions, θ_{Lj} and θ_{Tj} must add to unity.

In this section I assume that manufacturing is labor-intensive. It follows that labor's share in manufacturing must be greater than labor's share in food, and that the percentage of the labor force used in manufacturing must exceed the percentage of total land that is used in manufacturing. Let λ and θ be the notations for the matrices of coefficients shown in ([1.1], [2.1]) and ([3.1], [4.1]).

$$\lambda = \begin{pmatrix} \lambda_{LM} \lambda_{LF} \\ \lambda_{TM} \lambda_{TF} \end{pmatrix}, \qquad \theta = \begin{pmatrix} \theta_{LM} \theta_{TM} \\ \theta_{LF} \theta_{TF} \end{pmatrix}.$$

Since each row sum in λ and θ is unity, the determinants $|\lambda|$ and $|\theta|$ are given by

$$|\lambda| = \lambda_{LM} - \lambda_{TM},$$
$$|\theta| = \theta_{LM} - \theta_{LF},$$

and both $|\lambda|$ and $|\theta|$ are positive by the factor-intensity assumption.[5]

If coefficients of production are fixed, equations (1.1)–(4.1) are greatly simpli-

[4] This is the procedure used by Meade, *op. cit.* The λ and θ notation has been used by Amano, *op. cit.* Expressing small changes in relative or percentage terms is a natural procedure when technology exhibits constant returns to scale.

[5] Let P and W represent the diagonal matrices,

$$\begin{pmatrix} p_M & 0 \\ 0 & p_F \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w & 0 \\ 0 & r \end{pmatrix},$$

respectively, and E and X represent the diagonal matrices of factor endowments and commodity outputs. Then $\lambda = E^{-1}AX$ and $\theta = P^{-1}A'W$. Since $|A| > 0$ and the determinants of the four diagonal matrices are all positive, $|\lambda|$ and $|\theta|$ must be positive. This relation among the signs of $|\lambda|$, $|\theta|$, and $|A|$ is proved by Amano, *op. cit.*, and Akira Takayama, "On a Two-Sector Model of Economic Growth: A Comparative Statics Analysis," *Review of Economic Studies*, June, 1963.

fied as every a^*_{ij} and, therefore, the λ and θ weighted sums of the a^*_{ij}'s reduce to zero. In the case of variable coefficients, sufficient extra conditions to determine the a^*'s are easily derived. Consider, first, the maximizing role of the typical competitive entrepreneur. For any given level of output he attempts to minimize costs; that is he minimizes unit costs. In the manufacturing industry these are given by $(a_{LM}\ w + a_{TM}r)$. The entrepreneur treats factor prices as fixed, and varies the a's so as to set the derivative of costs equal to zero. Dividing by p_M and expressing changes in relative terms leads to equation (6). Equation (7) shows the corresponding relationship for the food industry.

$$\theta_{LM}a^*_{LM} + \theta_{TM}a^*_{TM} = 0\,, \qquad (6)$$

$$\theta_{LF}a^*_{LF} + \theta_{TF}a^*_{TF} = 0\,. \qquad (7)$$

With no technological change, alterations in factor proportions must balance out such that the θ-weighted average of the changes in input coefficients in each industry is zero.

This implies directly that the relationship between changes in factor prices and changes in commodity prices is *identical* in the variable and fixed coefficients cases, an example of the Wong-Viner envelope theorem. With costs per unit of output being minimized, the change in costs resulting from a small change in factor prices is the same whether or not factor proportions are altered. The saving in cost from such alterations is a second-order small.[6]

A similar kind of argument definitely does *not* apply to the λ-weighted average of the a^*'s for each factor that appears in the factor market-clearing relationships. For example, $(\lambda_{LM}a^*_{LM} + \lambda_{LF}a^*_{LF})$ shows the percentage change in the total quantity of labor required by the economy as a result of changing factor proportions in each industry at unchanged outputs. The crucial feature here is that if factor prices change, factor proportions alter in the same direction in both industries. The extent of this change obviously depends upon the elasticities of substitution between factors in each industry. In a competitive equilibrium (and with the internal tangencies implicit in earlier assumptions), the slope of the isoquant in each industry is equal to the ratio of factor prices. Therefore the elasticities of substitution can be defined as in (8) and (9):

$$\sigma_M = \frac{a^*_{TM} - a^*_{LM}}{w^* - r^*}, \qquad (8)$$

$$\sigma_F = \frac{a^*_{TF} - a^*_{LF}}{w^* - r^*}. \qquad (9)$$

Together with (6) and (7) a subset of four equations relating the a^*'s to the change in the relative factor prices is obtained. They can be solved in pairs; for example (6) and (8) yield solutions for the a^*'s of the M industry. In general,

$$a^*_{Lj} = -\theta_{Tj}\sigma_j(w^* - r^*); \qquad j = M, F\,.$$

$$a^*_{Tj} = \theta_{Lj}\sigma_j(w^* - r^*); \qquad j = M, F\,.$$

These solutions for the a^*'s can then be substituted into equations (1.1)–(4.1) to obtain

$$\lambda_{LM}M^* + \lambda_{LF}F^* = L^* + \delta_L(w^* - r^*)\,, \qquad (1.2)$$

$$\lambda_{TM}M^* + \lambda_{TF}F^* = T^* - \delta_T(w^* - r^*)\,, \qquad (2.2)$$

$$\theta_{LM}w^* + \theta_{TM}r^* = p^*_M\,, \qquad (3.2)$$

$$\theta_{LF}w^* + \theta_{TF}r^* = p^*_F\,, \qquad (4.2)$$

[6] For another example of the Wong-Viner theorem, for changes in real income along a transformation schedule, see Ronald W. Jones, "Stability Conditions in International Trade: A General Equilibrium Analysis," *International Economic Review*, May, 1961.

where $\delta_L = \lambda_{LM}\theta_{TM}\sigma_M + \lambda_{LF}\theta_{TF}\sigma_F$,

$$\delta_T = \lambda_{TM}\theta_{LM}\sigma_M + \lambda_{TF}\theta_{LF}\sigma_F .$$

In the fixed-coefficients case, δ_L and δ_T are zero. In general, δ_L is the aggregate percentage saving in labor inputs at unchanged outputs associated with a 1 per cent rise in the relative wage rate, the saving resulting from the adjustment to less labor-intensive techniques in both industries as relative wages rise.

The structure of the production model with variable coefficients is exhibited in equations (1.2)–(4.2). The latter pair states that factor prices are dependent only upon commodity prices, which is the factor-price equalization theorem.[7] If commodity prices are unchanged, factor prices are constant and equations (1.2) and (2.2) state that changes in commodity outputs are linked to changes in factor endowments via the λ matrix in precisely the same way as θ links factor price changes to commodity price changes. This is the basic duality feature of the production model.[8]

IV. THE MAGNIFICATION EFFECT

The nature of the link provided by λ or θ is revealed by examining the solutions for M^* and F^* at constant commodity prices in (1.2) and (2.2) and for w^* and r^* in equations (3.2) and (4.2).[9] If both endowments expand at the same rate, both commodity outputs expand at identical rates. But if factor endowments expand at different rates, the commodity intensive in the use of the fastest growing factor expands at a greater rate than either factor, and the other commodity grows (if at all) at a slower rate than either factor. For example, suppose labor expands more rapidly than land. With M labor-intensive,

$$M^* > L^* > T^* > F^* .$$

This *magnification effect* of factor endowments on commodity outputs at unchanged commodity prices is also a feature of the dual link between commodity and factor prices. In the absence of technological change or excise taxes or subsidies, if the price of M grows more rapidly than the price of F,

$$w^* > p_M^* > p_F^* > r^* .$$

Turned the other way around, the source of the magnification effect is easy to detect. For example, since the relative change in the price of either commodity is a positive weighted average of factor price changes, it must be bounded by these changes. Similarly, if input coefficients are fixed (as a consequence of assuming constant factor and commodity prices), any disparity in the growth of outputs is reduced when considering the consequent changes in the economy's demand for factors. The reason, of course, is that each good requires both factors of production.

Two special cases have been especially significant in the theory of international trade. Suppose the endowment of only one factor (say labor) rises. With L^* positive and T^* zero, M^* exceeds L^* and F^* is negative. This is the Rybczynski theorem in the theory of international

[7] Factor endowments come into their own in influencing factor prices if complete specialization is allowed (or if the number of factors exceeds the number of commodities). See Samuelson, "Prices of Factors and Goods in General Equilibrium," *Review of Economic Studies*, Vol. XXI, No. 1 (1953–54), for a detailed discussion of this issue.

[8] The reciprocal relationship between the effect of a rise in the price of commodity i on the return to factor j and the effect of an increase in the endowment of factor j on the output of commodity i is discussed briefly by Samuelson, *ibid.*

[9] The solutions, of course, are given by the elements of λ^{-1} and θ^{-1}. If M is labor-intensive, the diagonal elements of λ^{-1} and θ^{-1} are positive and exceed unity, while off-diagonal elements are negative.

trade: At unchanged commodity prices an expansion in one factor results in an absolute decline in the commodity intensive in the use of the other factor.[10] Its dual underlies the Stolper-Samuelson tariff theorem.[11] Suppose p_F^* is zero (for example, F could be taken as numeraire). Then an increase in the price of M (brought about, say, by a tariff on imports of M) raises the return to the factor used intensively in M by an even greater relative amount (and lowers the return to the other factor). In the case illustrated, the *real* return to labor has unambiguously risen.

For some purposes it is convenient to consider a slight variation of the Stolper-Samuelson theorem. Let p_j stand for the *market* price of j as before, but introduce a set of domestic excise taxes or subsidies so that $s_j p_j$ represents the price received by producers in industry j; s_j is one plus the ad valorem rate of subsidy to the industry.[12] The effect of an imposition of subsidies on factor prices is given in equations (3.3) and (4.3):

$$\theta_{LM} w^* + \theta_{TM} r^* = p_M^* + s_M^*, \quad (3.3)$$

$$\theta_{LF} w^* + \theta_{TF} r^* = p_F^* + s_F^*. \quad (4.3)$$

At fixed commodity prices, what impact does a set of subsidies have on factor prices? The answer is that all the subsidies are "shifted backward" to affect returns to factors in a *magnified* fashion. Thus, if M is labor-intensive and if the M industry should be especially favored by the subsidy,

$$w^* > s_M^* > s_F^* > r^*.$$

The *magnification* effect in this problem and its dual reflects the basic structure of the model with fixed commodity prices. However, if a demand relationship is introduced, prices are determined within the model and can be expected to adjust to a change in factor endowments or, in the dual problem, to a change in excise subsidies (or taxes). In the next section I discuss the feedback effect of these induced price changes on the composition of output and relative factor prices. The crucial question to be considered concerns the extent to which commodity price changes can dampen the initial magnification effects that are produced at constant prices.

V. THE EXTENDED MODEL: DEMAND ENDOGENOUS

To close the production model I assume that community taste patterns are homothetic and ignore any differences between the taste patterns of laborers and landlords. Thus the ratio of the quantities consumed of M and F depends only upon the relative commodity price ratio, as in equation (5).

$$\frac{M}{F} = f\left(\frac{p_M}{p_F}\right). \quad (5)$$

In terms of the rates of change, (5.1) serves to define the elasticity of substitution between the two commodities on the demand side, σ_D.

$$(M^* - F^*) = -\sigma_D(p_M^* - p_F^*). \quad (5.1)$$

[10] T. M. Rybczynski, "Factor Endowments and Relative Commodity Prices," *Economica*, November, 1955. See also Jones, "Factor Proportions and the Heckscher-Ohlin Theorem," *Review of Economic Studies*, October, 1956.

[11] W. F. Stolper and P. A. Samuelson, "Protection and Real Wages," *Review of Economic Studies*, November, 1941. A graphical analysis of the dual relationship between the Rybczynski theorem and the Stolper-Samuelson theorem is presented in Jones, "Duality in International Trade: A Geometrical Note," *Canadian Journal of Economics and Political Science*, August, 1965.

[12] I restrict the discussion to the case of excise subsidies because of the resemblance it bears to some aspects of technological change, which I discuss later. In the case of taxes, $s_j = 1/(1 + t_j)$ where t_j represents the ad valorem rate of excise tax.

The effect of a change in factor endowments at constant commodity prices was considered in the previous section. With the model closed by the demand relationship, commodity prices adjust so as to clear the commodity markets. Equation (5.1) shows directly the change in the ratio of outputs consumed. Subtracting (2.2) from (1.2) yields the change in the ratio of outputs produced.

$$(M^* - F^*) = \frac{1}{|\lambda|}(L^* - T^*) + \frac{(\delta_L + \delta_T)}{|\lambda|}(w^* - r^*).$$

The change in the factor price ratio (with no subsidies or taxes) is given by

$$(w^* - r^*) = \frac{1}{|\theta|}(p_M^* - p_F^*),$$

so that, by substitution,

$$(M^* - F^*) = \frac{1}{|\lambda|}(L^* - T^*) + \sigma_S(p_M^* - p_F^*),$$

where

$$\sigma_S = \frac{1}{|\lambda||\theta|}(\delta_L + \delta_T).$$

σ_S represents the elasticity of substitution between commodities on the *supply* side (along the transformation schedule).[13] The change in the commodity price ratio is then given by the mutual interaction of demand and supply:

$$(p_M^* - p_F^*) = -\frac{1}{|\lambda|(\sigma_S + \sigma_D)}(L^* - T^*). \quad (10)$$

Therefore the resulting change in the ratio of commodities produced is

$$(M^* - F^*) = \frac{1}{|\lambda|} \cdot \frac{\sigma_D}{\sigma_S + \sigma_D}(L^* - T^*). \quad (11)$$

With commodity prices adjusting to the initial output changes brought about by the change in factor endowments, the composition of outputs may, in the end, not change by as much, relatively, as the factor endowments. This clearly depends upon whether the "elasticity" expression, $\sigma_D/(\sigma_S + \sigma_D)$, is smaller than the "factor-intensity" expression, $|\lambda|$. Although it is *large* values of σ_S (and the underlying elasticities of factor substitution in each industry, σ_M and σ_F) that serve to dampen the spread of outputs, it is *small* values of σ_D that accomplish the same end. This comparison between elasticities on the demand and supply side is familiar to students of public finance concerned with questions of tax (or subsidy) incidence and shifting. I turn now to this problem.

The relationship between the change in factor prices and subsidies is given by (3.3) and (4.3). Solving for the change in the ratio of factor prices,

$$(w^* - r^*) = \frac{1}{|\theta|}\{(p_M^* - p_F^*) + (s_M^* - s_F^*)\}. \quad (12)$$

Consider factor endowments to be fixed. Any change in factor prices will nonethe-

[13] I have bypassed the solution for M^* and F^* separately given from (1.2) and (2.2). After substituting for the factor price ratio in terms of the commodity price ratio the expression for M^* could be written as

$$M^* = \frac{1}{|\lambda|}[\lambda_{TF}L^* - \lambda_{LF}T^*] + e_M(p_M^* - p_F^*),$$

where, e_M, the shorthand expression for $1/|\lambda||\theta|$ $(\lambda_{TF}\delta_L + \lambda_{LF}\delta_T)$, shows the percentage change in M that would be associated with a 1 per cent rise in M's relative price along a given transformation schedule. It is a "general equilibrium" elasticity of supply, as discussed in Jones, "Stability Conditions . . . ," *op. cit.* It is readily seen that $\sigma_S = e_M + e_F$. Furthermore, $\theta_M e_M = \theta_F e_F$, where θ_M and θ_F denote the share of each good in the national income.

less induce a readjustment of commodity outputs. On the supply side,

$$(M^* - F^*) = \sigma_S\{(p_M^* - p_F^*) + (s_M^* - s_F^*)\}.$$

The relative commodity price change that equates supply and demand is

$$(p_M^* - p_F^*) = -\frac{\sigma_S}{\sigma_S + \sigma_D}(s_M^* - s_F^*). \quad (13)$$

Substituting back into the expression for the change in the factor price ratio yields

$$(w^* - r^*) = \frac{1}{|\theta|}\cdot\frac{\sigma_D}{\sigma_S + \sigma_D}(s_M^* - s_F^*). \quad (14)$$

This is a familiar result. Suppose M is subsidized more heavily than F. Part of the subsidy is shifted backward, affecting relatively favorably the factor used intensively in the M-industry (labor). Whether labor's relative return expands by a greater proportion than the spread in subsidies depends upon how much of the subsidy has been passed forward to consumers in the form of a relatively lower price for M. And this, of course, depends upon the relative sizes of σ_S and σ_D.

Notice the similarity between expressions (11) and (14). Factors produce commodities, and a change in endowments must result in an altered composition of production, by a magnified amount at unchanged prices. By analogy, subsidies "produce" returns to factors, and a change in the pattern of subsidies alters the distribution of income. In each case, of course, the extent of readjustment required is eased if commodity prices change, by a factor depending upon the relative sizes of demand and supply elasticities of substitution.

VI. THE AGGREGATE ELASTICITY OF SUBSTITUTION

The analysis of a change in factor endowments leading up to equation (11) has a direct bearing on a recent issue in the neoclassical theory of economic growth. Before describing this issue it is useful to introduce yet another elasticity concept—that of an economy-wide elasticity of substitution between factors.[14] With no subsidies, the relationship between the change in the factor price ratio and the change in endowments can be derived from (10). Thus,

$$(w^* - r^*) = -\frac{1}{|\lambda|\,|\theta|\,(\sigma_S + \sigma_D)}(L^* - T^*). \quad (15)$$

By analogy with the elasticity of substitution in a particular sector, define σ as the percentage rise in the land/labor endowment ratio required to raise the wage/rent ratio by 1 per cent. Directly from (15),

$$\sigma = |\lambda|\ |\theta|(\sigma_S + \sigma_D).$$

But recall that σ_S is itself a composite of the two elasticities of substitution in each industry, σ_M and σ_F. Thus σ can be expressed in terms of the three *primary* elasticities of substitution in this model:

$$\sigma = Q_M\sigma_M + Q_F\sigma_F + Q_D\sigma_D,$$

$$\text{where } Q_M = \theta_{LM}\lambda_{TM} + \theta_{TM}\lambda_{LM},$$

$$Q_F = \theta_{LF}\lambda_{TF} + \theta_{TF}\lambda_{LF},$$

$$Q_D = |\lambda|\cdot|\theta|.$$

Note that σ is not just a linear expression in σ_M, σ_F, and σ_D—it is a weighted

[14] For previous uses see Amano, "Determinants of Comparative Costs: A Theoretical Approach," *Oxford Economic Papers*, November, 1964; and E. Drandakis, "Factor Substitution in the Two-Sector Growth Model," *Review of Economic Studies*, October, 1963.

average of these three elasticities as $\Sigma Q_i = 1$. Note also that σ can be positive even if the elasticity of substitution in each industry is zero, for it incorporates the effect of intercommodity substitution by consumers as well as direct intracommodity substitution between factors.

Finally, introduce the concept, σ, into expression (11) for output changes:

$$(M^* - F^*) = \frac{|\theta| \sigma_D}{\sigma}(L^* - T^*), \quad (11')$$

and into expression (14) for the change in factor prices in the subsidy case:

$$(w^* - r^*) = \frac{|\lambda| \sigma_D}{\sigma}(s_M^* - s_F^*). \quad (14')$$

One consequence is immediately apparent: If the elasticity of substitution between commodities on the part of consumers is no greater than the over-all elasticity of substitution between factors, the *magnification* effects discussed in Section IV are more than compensated for by the damping effect of price changes.

VII. CONVERGENCE TO BALANCED GROWTH

The two-sector model of production described in Sections I–VI can be used to analyze the process of economic growth. Already I have spoken of increases in factor endowments and the consequent "growth" of outputs. But a more satisfactory growth model would allow for the growth of at least one factor of production to be determined by the system rather than given parametrically. Let the factor "capital" replace "land" as the second factor in the two-sector model (replace T by K). And let M stand for machines rather than manufacturing goods. To simplify, I assume capital does not depreciate. The new feedback element in the system is that the rate of increase of the capital stock, K^*, depends on the current output of machines, M. Thus $K^* = M/K$. The "demand" for M now represents savings.

Suppose the rate of growth of the labor force, L^*, is constant. At any moment of time the rate of capital accumulation, K^*, either exceeds, equals, or falls short of L^*. Of special interest in the neoclassical theory of growth (with no technological progress) is the case of balanced growth where $L^* = K^*$. Balance in the growth of factors will, as we have seen, result in balanced growth as between the two commodities (at the same rate). But if L^* and K^* are not equal, it becomes necessary to inquire whether they tend toward equality (balanced growth) asymptotically or tend to diverge even further.

If machines are produced by labor-intensive techniques, the rate of growth of machines exceeds that of capital if labor is growing faster than capital, or falls short of capital if capital is growing faster than labor. (This is the result in Section IV, which is dampened, but not reversed, by the price changes discussed in Section V.) Thus the rate of capital accumulation, if different from the rate of growth of the labor supply, falls or rises toward it. The economy tends toward the balanced-growth path.

The difficulty arises if machines are capital intensive. If there is no price change, the change in the composition of outputs must be a magnified reflection of the spread in the growth rates of factors. Thus if capital is growing more rapidly than labor, machine output will expand at a greater rate than either factor, and this only serves to widen the spread between the rates of growth of capital and labor even further.[15] Once account is taken of price changes, however, the change in the composition of outputs may be sufficiently dampened to allow convergence to balanced growth despite the fact that machines are capital intensive.

Re-examine equation (11′), replacing T^* by K^* and recognizing that $|\theta|$ is negative if machines are capital intensive. If σ exceeds $-|\theta|\sigma_D$, on balance a dampening of the ratio of outputs as compared to factor endowments takes place. This suggests the critical condition that must be satisfied by σ, as compared with σ_D and $|\theta|$, in order to insure stability. But this is not precisely the condition required. Rather, stability hinges upon the *sign* of $(M^* - K^*)$ being opposite to that of $(K^* - L^*)$. There is a presumption that when $(M^* - F^*)$ is smaller than $(K^* - L^*)$ (assuming both are positive) the output of the machine sector is growing less rapidly than is the capital stock. But the correspondence is not exact.

To derive the relationship between $(M^* - K^*)$ and $(M^* - F^*)$ consider the two ways of expressing changes in the national income (Y). It can be viewed as the sum of returns to factors or the sum of the values of output in the two sectors. Let θ_i refer to the share of factor i or commodity i in the national income. In terms of rates of change,

$$Y^* = \theta_L(w^* + L^*) + \theta_K(r^* + K^*) = \theta_M(p_M^* + M^*) + \theta_F(p_F^* + F^*) .$$

But the share of a factor in the national income must be an average of its share in each sector, with the weights given by the share of that sector in the national income. This, and equations (3.2) and (4.2), guarantee that

$$\theta_L w^* + \theta_K r^* = \theta_M p_M^* + \theta_F p_F^* .$$

That is, the rates of change of the financial components in the two expressions for Y^* balance, leaving an equality between the physical terms:

$$\theta_L L^* + \theta_K K^* = \theta_M M^* + \theta_F F^* .$$

The desired relationship is obtained by observing that θ_K equals $(1 - \theta_L)$ and θ_M is $(1 - \theta_F)$. Thus

$$(M^* - K^*) = \theta_F(M^* - F^*) - \theta_L(K^* - L^*) .$$

With this in hand it is easy to see that (from [11′]) $(M^* - K^*)$ is given by

$$(M^* - K^*) = \frac{\theta_L}{\sigma} \times \left\{ -\frac{\theta_F |\theta|}{\theta_L} \sigma_D - \sigma \right\} (K^* - L^*). \tag{16}$$

It is not enough for σ to exceed $-|\theta|\sigma_D$, it must exceed $-(\theta_F/\theta_L)|\theta|\sigma_D$ for convergence to balanced growth.[16] It nonetheless remains the case that σ greater than σ_D is sufficient to insure that the expression in brackets in (16) is negative. For (16) can be rewritten as (16′):

$$(M^* - K^*) = -\frac{\theta_L}{\sigma} \times \left\{ \sigma - \left[1 - \frac{\theta_{LM}}{\theta_L} \right] \sigma_D \right\} (K^* - L^*). \tag{16′}$$

[15] See Y. Shinkai, "On Equilibrium Growth of Capital and Labor," *International Economic Review*, May, 1960, for a discussion of the fixed-coefficients case. At constant commodity prices the impact of endowment changes on the composition of output is the same regardless of elasticities of substitution in production. Thus a necessary and sufficient condition in Shinkai's case is the factor-intensity condition. For the variable coefficients case the factor-intensity condition was first discussed by Hirofumi Uzawa, "On a Two-Sector Model of Economic Growth," *Review of Economic Studies*, October, 1961.

[16] The two requirements are equivalent if $\theta_F = \theta_L$, that is, if total consumption ($p_F F$) is matched exactly by the total wages (wL). This equality is made a basic assumption as to savings behavior in some models, where laborers consume all and capitalists save all. For example, see Uzawa, *ibid.*

Thus it is overly strong to require that σ exceed σ_D.[17]

VIII. SAVINGS BEHAVIOR

A popular assumption about savings behavior in the literature on growth theory is that aggregate savings form a constant percentage of the national income.[18] This, of course, implies that σ_D is unity. In this case it becomes legitimate to inquire as to the values of σ or σ_M and σ_F as compared with unity. For example, if each sector's production function is Cobb-Douglas (σ_M and σ_F each unity), stability is guaranteed. But the value "unity" that has a crucial role in this comparison only serves as a proxy for σ_D. With high σ_D even greater values for σ_M and σ_F (and σ) would be required.

If σ_D is unity when the savings ratio is constant, is its value higher or lower than unity when the savings ratio depends positively on the rate of profit? It turns out that this depends upon the technology in such a way as to encourage convergence to balanced growth precisely in those cases where factor intensities are such as to leave it in doubt.

The capital goods, machines, are demanded not for the utility they yield directly, but for the stream of additional future consumption they allow. This is represented by the rate of return (or profit), which is linked by the technology to the relative price of machines according to the magnification effects implicit in the Stolper-Samuelson theorem. The assumption that the savings ratio (the fraction of income devoted to new machines) rises as the rate of profit rises implies that the savings ratio rises as the relative price of machines rises (and thus that σ_D is less than unity) if and only if machines are capital intensive. Of course the savings assumption also implies that σ_D exceeds unity (that is, that the savings ratio falls as the relative price of machines rises) if machines are labor intensive, but convergence to balanced growth is already assured in this case.[19]

IX. THE ANALYSIS OF TECHNOLOGICAL CHANGE

The preceding sections have dealt with the structure of the two-sector model of production with a given technology. They nonetheless contain the ingredients necessary for an analysis of the effects of technological progress. In this concluding section I examine this problem and simplify by assuming that factor endowments remain unchanged and subsidies are zero. I concentrate on the impact of a change in production conditions on relative prices. The effect on outputs is considered implicitly in deriving the price changes.

Consider a typical input coefficient, a_{ij}, as depending both upon relative factor prices and the state of technology:

$$a_{ij} = a_{ij}\left(\frac{w}{r},\ t\right).$$

In terms of the relative rates of change, a^*_{ij} may be decomposed as

$$a^*_{ij} = c^*_{ij} - b^*_{ij}\,.$$

c^*_{ij} denotes the relative change in the input-output coefficient that is called forth by a change in factor prices as of a given technology. The b^*_{ij} is a measure of

[17] A condition similar to (16′), with the assumption that $\sigma_D = 1$, is presented by Amano, "A Two-Sector Model of Economic Growth Involving Technical Progress" (unpublished).

[18] For example, see Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, February, 1956.

[19] For a more complete discussion of savings behavior as related to the rate of profit, see Uzawa, "On a Two-Sector Model of Economic Growth: II," *Review of Economic Studies*, June, 1963; and Kenichi Inada, "On Neoclassical Models of Economic Growth," *Review of Economic Studies*, April, 1965.

technological change that shows the alteration in a_{ij} that would take place at constant factor prices. Since technological progress usually involves a *reduction* in the input requirements, I define b_{ij}^* as $-1/a_{ij} \cdot \partial a_{ij}/\partial t$.

The b_{ij}^* are the basic expressions of technological change. After Section III's discussion, it is not surprising that it is the λ and θ weighted averages of the b_{ij}^* that turn out to be important. These are defined by the following set of π's:

$$\pi_j = \theta_{Lj}b_{Lj}^* + \theta_{Tj}b_{Tj}^* \qquad (j = M, F),$$

$$\pi_i = \lambda_{iM}b_{iM}^* + \lambda_{iF}b_{iF}^* \qquad (i = L, T).$$

If a B^* matrix is defined in a manner similar to the original A matrix, π_M and π_F are the sums of the elements in each column weighted by the relative factor shares, and π_L and π_T are sums of the elements in each row of B^* weighted by the fractions of the total factor supplies used in each industry. Thus π_M, assumed non-negative, is a measure of the rate of technological advance in the M-industry and π_L, also assumed non-negative, reflects the over-all labor-saving feature of technological change.

Turn now to the equations of change. The c_{ij}^* are precisely the a_{ij}^* used in equations (6)–(9) of the model without technological change. This subset can be solved, just as before, for the response of input coefficients to factor price changes. After substitution, the first four equations of change (equations [1.1]–[4.1]) become

$$\lambda_{LM}M^* + \lambda_{LF}F^* = \pi_L + \delta_L(w^* - r^*), \qquad (1.4)$$

$$\lambda_{TM}M^* + \lambda_{TF}F^* = \pi_T - \delta_T(w^* - r^*), \qquad (2.4)$$

$$\theta_{LM}w^* + \theta_{TM}r^* = p_M^* + \pi_M, \qquad (3.4)$$

$$\theta_{LF}w^* + \theta_{TF}r^* = p_F^* + \pi_F. \qquad (4.4)$$

The parameters of technological change appear only in the first four relationships and enter there in a particularly simple form. In the first two equations it is readily seen that, in part, technological change, through its impact in reducing input coefficients, has precisely the same effects on the system as would a change in factor endowments. π_L and π_T replace L^* and T^* respectively. In the second pair of equations the improvements in industry outputs attributable to technological progress enter the model precisely as do industry subsidies in equations (3.3) and (4.3) of Section IV. Any general change in technology or in the quality of factors (that gets translated into a change in input coefficiencies) has an impact on prices and outputs that can be decomposed into the two kinds of parametric changes analyzed in the preceding sections.

Consider the effect of progress upon relative commodity and factor prices. The relationship between the changes in the two sets of prices is the same as in the subsidy case (see equation [12]):

$$(w^* - r^*) = \frac{1}{|\theta|}\{(p_M^* - p_F^*) + (\pi_M - \pi_F)\}. \qquad (17)$$

Solving separately for each relative price change,

$$(p_M^* - p_F^*) = -\frac{|\theta|}{\sigma} \times \{(\pi_L - \pi_T) + |\lambda|\sigma_S(\pi_M - \pi_F)\}, \qquad (18)$$

$$(w^* - r^*) = -\frac{1}{\sigma} \times \{(\pi_L - \pi_T) - |\lambda|\sigma_D(\pi_M - \pi_F)\}. \qquad (19)$$

For convenience I refer to $(\pi_L - \pi_T)$ as the "differential factor effect" and $(\pi_M -$

π_F) as the "differential industry effect."[20]

Define a change in technology as "regular" if the differential factor and industry effects have the same sign.[21] For example, a change in technology that is relatively "labor-saving" for the economy as a whole ($[\pi_L - \pi_T]$ positive) is considered "regular" if it also reflects a relatively greater improvement in productivity in the labor-intensive industry. Suppose this to be the case. Both effects tend to depress the relative price of commodity M: The "labor-saving" feature of the change works exactly as would a relative increase in the labor endowment to reduce the relative price of the labor-intensive commodity (M). And part of the differential industry effect, like a relative subsidy to M, is shifted forward in a lower price for M.

Whereas the two components of "regular" technological change reinforce each other in their effect on the commodity price ratio, they pull the factor price ratio in opposite directions. The differential factor effect in the above case serves to depress the wage/rent ratio. But part of the relatively greater improvement in the labor-intensive M industry is shifted backward to increase, relatively, the return to labor. This "backward" shift is more pronounced the greater is the elasticity of substitution on the demand side. There will be some "critical" value of σ_D, above which relative wages will rise despite the downward pull of the differential factor effect:

$$(w^* - r^*) > 0 \text{ if and only if } \sigma_D > \frac{(\pi_L - \pi_T)}{|\lambda|(\pi_M - \pi_F)}.$$

If technological progress is not "regular," these conclusions are reversed. Suppose $(\pi_L - \pi_T) > 0$, but nonetheless $(\pi_M - \pi_F) < 0$. This might be the result, say, of technological change where the primary impact is to reduce labor requirements in food production. Labor is now affected relatively adversely on both counts, the differential factor effect serving to depress wages as before, and the differential industry effect working to the relative advantage of the factor used intensively in food production, land. On the other hand, the change in relative commodity prices is now less predictable. The differential factor effect, in tending to reduce M's relative price, is working counter to the differential industry effect, whereby the F industry is experiencing more rapid technological advance. The differential industry effect will, in this case, dominate if the elasticity of substitution between goods on the supply side is high enough.

$$(p_M^* - p_F^*) > 0 \text{ if and only if } \sigma_S > -\frac{(\pi_L - \pi_T)}{|\lambda|(\pi_M - \pi_F)}.$$

The differential factor and industry effects are not independent of each other.

[20] The suggestion that a change in technology in a particular industry has both "factor-saving" and "cost-reducing" aspects has been made before. See, for example, J. Bhagwati and H. Johnson, "Notes on Some Controversies in the Theory of International Trade," *Economic Journal*, March, 1960; and G. M. Meier, *International Trade and Development* (New York: Harper & Row, 1963), chap. i. Contrary to what is usually implied, I point out that a Hicksian "neutral" technological change in one or more industries has, nonetheless, a "factor-saving" or "differential factor" effect. The problem of technological change has been analyzed in numerous articles; perhaps those by H. Johnson, "Economic Expansion and International Trade," *Manchester School of Economic and Social Studies*, May, 1955; and R. Findlay and H. Grubert, "Factor Intensities, Technological Progress and the Terms of Trade," *Oxford Economic Papers*, February, 1959, should especially be mentioned.

[21] Strictly speaking, I want to allow for the possibility that one or both effects are zero. Thus technological change is "regular" if and only if $(\pi_L - \pi_T)(\pi_M - \pi_F) \geqq 0$.

Some insight into the nature of the relationship between the two can be obtained by considering two special cases of "neutrality."

Suppose, first, that technological change is "Hicksian neutral" in each industry, implying that, at unchanged factor prices, factor proportions used in that industry do not change.[22] In terms of the B^* matrix, the rows are identical ($b^*_{Lj} = b^*_{Tj}$). As can easily be verified from the definition of the π's, in this case

$$(\pi_L - \pi_T) = |\lambda|(\pi_M - \pi_F),$$

and technological change must be "regular." If, over-all, technological change is "labor-saving" (and note that this can happen even if it is Hicksian neutral in each industry), the price of the relatively labor-intensive commodity must fall. Relative wages will, nonetheless, rise if σ_D exceeds the critical value shown earlier, which in this case reduces to unity.

The symmetrical nature of this approach to technological change suggests an alternative definition of neutrality, in which the columns of the B^* matrix are equal. This type of neutrality indicates that input requirements for any factor, i, have been reduced by the same relative amount in every industry. The relationship between the differential factor and industry effects is given by

$$(\pi_M - \pi_F) = |\theta|(\pi_L - \pi_T).$$

Again, technological change must be "regular." If the reduction in labor coefficients in each industry exceeds the reduction in land coefficients, this must filter through (in dampened form unless each industry uses just one factor) to affect relatively favorably the labor-intensive industry. The remarks made in the case of Hicksian neutrality carry over to this case, except for the fact that the critical value which σ_D must exceed in order for the differential industry effect to outweigh the factor effect on relative wages now becomes higher. Specifically, σ_D must exceed $1/|\lambda||\theta|$, which may be considerably greater than unity. This reflects the fact that in the case of Hicksian neutrality $(\pi_L - \pi_T)$ is smaller than $(\pi_M - \pi_F)$, whereas the reverse is true in the present case.

With Hicksian neutrality the paramount feature is the difference between rates of technological advance in each industry. This spills over into a differential factor effect only because the industries require the two factors in differing proportions. With the other kind of neutrality the basic change is that the input requirements of one factor are cut more than for the other factor. As we have just seen, this is transformed into a differential industry effect only in dampened form.

These cases of neutrality are special cases of "regular" technological progress. The general relationship between the differential factor and industry effects can be derived from the definitions to yield

$$(\pi_L - \pi_T) = Q_M\beta_M + Q_F\beta_F + |\lambda|(\pi_M - \pi_F), \quad (20)$$

and

$$(\pi_M - \pi_F) = Q_L\beta_L + Q_T\beta_T + |\theta|(\pi_L - \pi_T). \quad (21)$$

In the first equation the differential factor effect is broken down into three components: the labor-saving bias of technical change in each industry (β_j is defined as $b^*_{Lj} - b^*_{Tj}$) and the differential industry effect.[23] In the second expres-

[22] See Hicks, *The Theory of Wages* (New York: Macmillan Co., 1932).

[23] Note that Q_M and Q_F are the same weights as those defined in Section VI. The analogy between

sion the differential industry effect is shown as a combination of the relatively greater saving in each factor in the M industry (β_L, for example, is $b^*_{LM} - b^*_{LF}$) and the differential factor effect.[24] With these relationships at hand it is easy to see how it is the possible asymmetry between the row elements and/or the column elements of the B^* matrix that could disrupt the "regularity" feature of technical progress.[25]

For some purposes it is useful to make the substitution from either (20) or (21) into the expressions for the changes in relative factor and commodity prices shown by (17)–(19). For example, if technological change is "neutral" in the sense described earlier, where the reduction in the input coefficient is the same in each industry (although different for each factor), β_L and β_T are zero in (21) and the relationship in (17) can be rewritten as

$$(w^* - r^*) = \frac{1}{|\theta|}(p^*_M - p^*_F) + (\pi_L - \pi_T).$$

To make things simple, suppose π_T is zero. The uniform reduction in labor input coefficients across industries might reflect, say, an improvement in labor quality attributable to education. Aside from the effect of any change in commodity prices on factor prices (of the Stolper-Samuelson variety), relative wages are directly increased by the improvement in labor quality.

Alternatively, consider substituting (20) into (19), to yield (19′):

$$(w^* - r^*) = -\frac{1}{\sigma}\left\{Q_M\beta_M + Q_F\beta_F + Q_D(1 - \sigma_D)\frac{(\pi_M - \pi_F)}{|\theta|}\right\}. \tag{19'}$$

Will technological change that is Hicks neutral in every industry leave the factor price ratio unaltered at a given ratio of factor endowments? Equation (19′) suggests a negative answer to this query unless progress is at the same rate in the two industries ($\pi_M = \pi_F$) or unless σ_D is unity.[26]

There exists an extensive literature in the theory of international trade concerned with (*a*) the effects of differences in production functions on pre-trade factor and commodity price ratios (and thus on positions of comparative advantage), and (*b*) the impact of growth (in factor supplies) or changes in technological knowledge in one or more countries on the world terms of trade.[27] The analysis of this paper is well suited to the discus-

the composition of σ and that of $(\pi_L - \pi_T)$ becomes more apparent if $|\lambda|\,(\pi_M - \pi_F)$ is rewritten as $Q_D \cdot \{(\pi_M - \pi_F)/|\theta|\}$. The differential factor effect is a weighted average of the Hicksian factor biases in each industry and a magnified $(1/|\theta|)$ differential industry effect.

[24] Q_L equals $(\lambda_{LF}\theta_{LM} + \lambda_{LM}\theta_{LF})$, and Q_T is $(\lambda_{TF}\theta_{TM} + \lambda_{TM}\theta_{TF})$. Note that $Q_L + Q_T$ equals $Q_M + Q_F$.

[25] These relationships involve the *difference* between π_L and π_T, on the one hand, and π_M and π_F on the other. Another relationship involving *sums* of these terms is suggested by the national income relationship, as discussed in Section VII. With technical progress. $\theta_M\pi_M + \theta_F\pi_F$ equals $\theta_L\pi_L + \theta_T\pi_T$.

[26] Recalling n. 23, consider the following question: If the elasticity of substitution between factors is unity in every sector, will a change in the ratio of factor endowments result in an equal percentage change in the factor price ratio? From Section VI it is seen that this result can be expected only if σ_D is unity.

[27] See H. Johnson, "Economic Development and International Trade," *Money, Trade, and Economic Growth* (London: George Allen & Unwin, 1962), chap. iv, and the extensive bibliography there listed. The most complete treatment of the effects of various differences in production conditions on positions of comparative advantage is given by Amano, "Determinants of Comparative Costs . . . ," *op. cit.*, who also discusses special cases of Harrod neutrality. For a recent analysis of the impact of endowment and technology changes on the terms of trade see Takayama, "Economic Growth and International Trade," *Review of Economic Studies*, June, 1964.

sion of these problems. The connection between (*a*) and expressions (17)–(19) is obvious. For (*b*) it is helpful to observe that the impact of any of these changes on world terms of trade depends upon the effect in each country separately of these changes on production and consumption at constant commodity prices. The production effects can be derived from the four equations of change for the production sector (equations [1.1]–[4.1] or later versions) and the consumption changes from equation (5.1).[28] The purpose of this paper is not to reproduce the results in detail but rather to expose those features of the model which bear upon all of these questions.

[28] Account must be taken, however, of the fact that with trade the quantities of M and F produced differ from the amounts consumed by the quantity of exports and imports.