□

A2.18 Let $f(\mathbf{x})$ be a real-valued function defined on R^n_+, and consider the matrix

$$\mathbf{H}^* \equiv \begin{pmatrix} 0 & f_1 & \dots & f_n \\ f_1 & f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n1} & \dots & f_{nn} \end{pmatrix}.$$

This is a different sort of *bordered Hessian* than we considered in the text. Here, the matrix of second-order partials is bordered by the first-order partials and a zero to complete the square matrix. The principal minors of this matrix are the determinants

$$D_2 = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix}, \qquad D_3 = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix}, \quad \dots \quad D_n = |\mathbf{H}^*|.$$

Arrow and Enthoven (1961) use the sign pattern of these principal minors to establish the following useful results:

i. If $f(\mathbf{x})$ is *quasiconcave*, these principal minors alternate in sign as follows: $D_2 \leq 0$, $D_3 \geq 0, \dots$.

ii. If for all $\mathbf{x} \geq 0$, these principal minors (which depend on $\mathbf{x}$) alternate in sign beginning with *strictly* negative: $D_2 < 0$, $D_3 > 0, \dots$, then $f(\mathbf{x})$ is quasiconcave on the non-negative orthant. Further, it can be shown that if, for all $\mathbf{x} \gg \mathbf{0}$, we have this same alternating sign pattern on those principal minors, then $f(\mathbf{x})$ is *strictly* quasiconcave on the (strictly) positive orthant.

(a) The function $f(x_1, x_2) = x_1 x_2 + x_1$ is quasiconcave on R^2_+. Verify that its principal minors alternate in sign as in (*ii*).

(b) Let $f(x_1, x_2) = a \ln(x_1 + x_2) + b$, where $a > 0$. Is this function strictly quasiconcave for $\mathbf{x} \gg \mathbf{0}$? Is it quasiconcave? How about for $\mathbf{x} \geq \mathbf{0}$? Justify.