VASSAR COLLEGE
POUGHKEEPSIE · NEW YORK 12604
Department of Economics

GEOFFREY A. JEHLE
jehle@vassar.edu

PHONE: 845-437-5210
FAX: 845-437-7576

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On Exercise 3.15, part (b), as corrected.

Recall the corrected problem asks you to show that for the CES form $y = \left(\sum_{i=1}^{n} \alpha_i x_i^\rho\right)^{1/\rho}$, where $\sum_{i=1}^{n} \alpha_i = 1$ and $0 \neq \rho < 1$, we will have $\lim_{\rho \to -\infty} y = \min\{x_1, \ldots, x_n\}$.

It may be counter-intuitive, as Bettina has noted, but it is nonetheless true that the limiting form of this CES form as the parameter $\rho \to -\infty$ is the simple (unweighted) Leontief form. To see this most clearly, let's just take the two input case. The arguments generalize straightforwardly. We begin by noting the followng:

Lemma For $(x_1, x_2) \geq (0, 0)$ and $0 \leq \alpha \leq 1$, $\lim_{r \to \infty} [\alpha x_1^\rho + (1-\alpha) x_2^\rho]^{1/\rho} = \max\{x_1, x_2\}$.

Proof: Suppose WLOG that $x_1 = \max\{x_1, x_2\}$. Then for $0 \leq \alpha \leq 1$ and $\rho \geq 1$,

$$\begin{aligned}
\alpha^{1/\rho} x_1 &\leq \alpha x_1 \\
&= [\alpha^\rho x_1^\rho]^{1/\rho} \\
&\leq [\alpha x_1^\rho]^{1/\rho} \\
&\leq [\alpha x_1^\rho + (1-\alpha) x_2^\rho]^{1/\rho} \\
&\leq [x_1^\rho]^{1/\rho} \\
&= x_1.
\end{aligned}$$

Using the first, fourth and final lines we may write

$$\alpha^{1/\rho} x_1 \leq [\alpha x_1^\rho + (1-\alpha) x_2^\rho]^{1/\rho} \leq x_1.$$

Taking the limit in this expression as $\rho \to \infty$ establishes the claim in the lemma. ∎

Now note that

$$\begin{aligned}
\lim_{r \to -\infty} [\alpha x_1^\rho + (1-\alpha) x_2^\rho]^{1/\rho} &= \lim_{r \to \infty} \left[\alpha x_1^{-\rho} + (1-\alpha) x_2^{-\rho}\right]^{-1/\rho} \\
&= \lim_{r \to \infty} \frac{1}{\left[\alpha \left(\frac{1}{x_1}\right)^\rho + (1-\alpha) \left(\frac{1}{x_2}\right)^\rho\right]^{1/\rho}} \\
&= \frac{1}{\max\{\frac{1}{x_1}, \frac{1}{x_2}\}} \\
&= \min\{x_1, x_2\}.
\end{aligned}$$

where the second to last line follows from the lemma, above, establishing the proposition.