Exercise 1.56

(a) The Stone-Geary utility function is $U(\mathbf{x}) = \Pi\ (x_j - a_j)^{bj}$ where $\Sigma\ b_j = 1$.

We want the expenditure function so we minimize expenditure subject to the utility level being at least u.

The Lagrangian is $L = \Sigma\ p_i x_i - \lambda\ [\Pi(x_j - a_j)^{bj}\ - u]$.

The first order conditions are

$p_i - \lambda b_i(x_i - a_i)^{-1}\Pi\ (x_j - a_j)^{bj}\ = 0$ all i = 1,2,…,n (FOC1)

and $\Pi\ (x_j - a_j)^{bj}\ = u$. (FOC2)

Both x_i and λ are functions of **p** and u.

The first order conditions give $p_i x_i - p_i a_i = \lambda b_i u$ all i = 1,2,…,n. Equation (1)

So $\lambda u = (p_i / b_i)(x_i - a_i)$ for all i =1,2,…,n.

Hence $x_j - a_j = (p_i / b_i)(x_i - a_i)(b_j / p_j)$ all i, j = 1,2,…,n as λu is independent from i.

Substitute this in FOC2.

So $u = \Pi\ [(p_i / b_i)(x_i - a_i)(b_j / p_j)]^{bj}$ where the product is over j = 1,2,…,n

$= \Pi\ [(p_i / b_i)(x_i - a_i)]^{bj}\ \Pi\ (b_j / p_j)]^{bj}$

$= (p_i / b_i)(x_i - ai)\ \Pi(b_j / p_j)^{bj}$ using $\Sigma\ b_j = 1$.

It follows that $p_i x_i = p_i a_i + b_i u\ \Pi(p_j / b_j)^{bj}$ Equation (2)

Summing equation (2) over i and using $\Sigma\ b_i = 1$, gives

$e(\mathbf{p},u) = \Sigma\ p_i x_i\ = \Sigma\ p_i a_i\ + u\ \Pi(p_j / b_j)^{bj}$ as the expenditure function for the Stone-Geary utility function.

Since $e(\mathbf{p},u) = y$, it follows that $v(\mathbf{p},y) = (y - \Sigma\ p_i a_i\ )\ \Pi\ (b_j / p_j)^{bj}$ is the indirect utility function for the Stone-Geary utility function.

(b) The first part of the question concerns the Hicksian demands $x_i(\mathbf{p},u)$ but this part is about the Marshallian demands $x^*_i(\mathbf{p},y)$. There at least three ways to go about answering this part. The first way is to maximize $U(\mathbf{x})$ subject to the budget constraint. The second way is to use the equality between $x_i(\mathbf{p},u)$ and $x^*_i(\mathbf{p},y)$ when $y = e(\mathbf{p},u)$, sum over Equation (1) to get $\Sigma\, p_i x^*_i - \Sigma\, p_i a_i = \Sigma\, \lambda b_i u = \lambda u$ and then substitute back into Equation (1) to get the result.

The posh way is to use Roy's Identity.

We have $\delta v(\mathbf{p},y)/\delta y = \Pi\, (b_j / p_j)^{bj}$

and $\delta v(\mathbf{p},y)/\delta p_i = [\, - (b_i / p_i)\, y - a_i + (b_i / p_i)\Sigma\, p_i a_i\,]\ \Pi\, (b_j / p_j)^{bj}$.

By Roy's Identity, $x^*_i(\mathbf{p},y) = - \delta v(\mathbf{p},y)/\delta p_i / \delta v(\mathbf{p},y)/\delta y$ and the result

$p_i x^*_i(\mathbf{p},y) = p_i a_i + b_i\, (y - \Sigma\, p_i a_i)$ follows.

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