2. Homework, Part I, Econ 973,
Department of Economics
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2. Homework Solutions

A2.1

(e) $f(x) = [\frac{3x}{x^3+1}]^2$.

Then, $f'(x) = 2\left[3\frac{x}{x^3+1}\right]\left[\frac{3}{x^3+1} - 9\frac{x^3}{(x^3+1)^2}\right] = 18\left[\frac{x}{x^3+1}\right]\left[\frac{1-2x^3}{(x^3+1)^2}\right] = 18\frac{x-2x^4}{(x^3+1)^3}$ and $f'(2) = -\frac{20}{27} < 0$.

Hence f is decreasing at $x = 2$.

Furthermore, $f''(x) = 18\frac{1-8x^3}{(x^3+1)^3} - 162x^2\frac{2-2x^4}{(x^3+1)^4}$ and $f''(2) = \frac{38}{27} > 0$.

Thus, f is locally convex at $x = 2$.

(f) $f(x) = [(\frac{1}{x^2} + 2) - (\frac{1}{x} - 2)]^4$.

Then, $f'(x) = 4\left[\frac{1}{x^2} + 4 - \frac{1}{x}\right]^3\left[-\frac{2}{x^3} + \frac{1}{x^2}\right]$ and $f'(2) = 0$.

Hence f is constant at $x = 2$.

Furthermore, $f''(x) = 12\left[\frac{1}{x^2} + 4 - \frac{1}{x}\right]^2\left[-\frac{2}{x^3} + \frac{1}{x^2}\right]^2 + 4\left[\frac{1}{x^2} + 4 - \frac{1}{x}\right]^3\left[\frac{6}{x^4} - \frac{2}{x^3}\right]$ and $f''(2) = \frac{1}{2}(\frac{15}{4})^3 > 0$.

Thus, f is locally convex at $x = 2$.

(g) $f(x) = \int_x^1 e^{t^2}dt = \frac{1}{2}e - \frac{1}{2x}e^{x^2}$.

Then, $f'(x) = -e^{x^2}$ and $f'(2) = -e^4 < 0$.

Hence f is decreasing at $x = 2$.

Furthermore, $f''(x) = -2xe^{x^2}$ and $f''(x) = -4e^4 < 0$.

Thus, f is locally concave at $x = 2$.

A2.2

(e) $f(x_1, x_2) = x_1^3 - 6x_1x_2 + x_2^3$.

Then, $\frac{\partial}{\partial x_1}f(x_1, x_2) = 3x_1^2 - 6x_2$ and $\frac{\partial}{\partial x_2}f(x_1, x_2) = 3x_2^2 - 6x_1$.

(f) $f(x_1, x_2) = 3x_1^2 - x_1x_2 + x_2$.

Then, $\frac{\partial}{\partial x_1} f(x_1, x_2) = 6x_1 - x_2$ and $\frac{\partial}{\partial x_2} f(x_1, x_2) = 1 - x_1$.

(g) $g(x_1, x_2, x_3) = \ln(x_1^2 - x_2x_3 - x_3^2)$. Then, $\frac{\partial}{\partial x_1} g(x_1, x_2, x_3) = \frac{2x_1}{x_1^2 - x_2x_3 - x_3^2}$,

$\frac{\partial}{\partial x_2} g(x_1, x_2, x_3) = \frac{-x_3}{x_1^2 - x_2x_3 - x_3^2}$, and $\frac{\partial}{\partial x_3} g(x_1, x_2, x_3) = \frac{-x_2 - 2x_3}{x_1^2 - x_2x_3 - x_3^2}$.

A2.4 Let $y = x_1^2x_2 + x_2^2x_3 + x_3^2x_1$. Then,

$$\begin{aligned}
\frac{\partial y}{\partial x_1} &= 2x_1x_2 + x_3^2, \\
\frac{\partial y}{\partial x_2} &= x_1^2 + 2x_2x_3, \\
\frac{\partial y}{\partial x_3} &= x_2^2 + 2x_3x_1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} + \frac{\partial y}{\partial x_3} &= 2x_1x_2 + x_3^2 + x_1^2 + 2x_2x_3 + x_2^2 + 2x_3x_1 \\
&= (x_1 + x_2)^2 + 2(x_2x_3 + x_3x_1) + x_3^2 \\
&= (x_1 + x_2 + x_3)^2.
\end{aligned}$$

2.8 $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$.

(a) Let $t > 0$. Then, $f(tx_1, tx_2) = \sqrt{t^2x_1^2 + t^2x_2^2} = t\sqrt{x_1^2 + x_2^2} = tf(x_1, x_2)$.

Hence, f is homogeneous of degree 1.

(b) $\frac{\partial}{\partial x_1} f(x_1, x_2) = \frac{1}{2} \frac{2x_1}{\sqrt{x_1^2 + x_2^2}}$ and $\frac{\partial}{\partial x_2} f(x_1, x_2) = \frac{1}{2} \frac{2x_2}{\sqrt{x_1^2 + x_2^2}}$.

Hence, as predicted by Euler's Theorem,

$$\begin{aligned}
\left(\frac{\partial}{\partial x_1} f(x_1, x_2)\right) x_1 + \left(\frac{\partial}{\partial x_2} f(x_1, x_2)\right) x_2 &= \frac{1}{2} \frac{2x_1}{\sqrt{x_1^2 + x_2^2}} x_1 + \frac{1}{2} \frac{2x_2}{\sqrt{x_1^2 + x_2^2}} x_2 \\
&= \frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2}} \\
&= \sqrt{x_1^2 + x_2^2} \\
&= f(x_1, x_2).
\end{aligned}$$

2.10 Let $h : D \to \mathbb{R}$ be homothetic. By the definition of homotheticity, $h(x) = g(f(x))$ where g is strictly increasing. We have to show that for all $x \in D$ and all $i, j \in \{1, \ldots, n\}$

$$\frac{\partial h(tx)/\partial x_i}{\partial h(tx)/\partial x_j}$$

is constant in $t > 0$.

$$\frac{\partial h(tx)}{\partial x_i} = \frac{\partial g(f(tx))}{\partial x_i} \overset{\text{lin. Homogeneity}}{=} \frac{\partial g(tf(x))}{\partial x_i} \overset{\text{Chain Rule}}{=} g'(tf(x))t\frac{\partial f(x)}{\partial x_i}$$

and

$$\frac{\partial h(tx)}{\partial x_j} = g'(tf(x))t\frac{\partial f(x))}{\partial x_j}.$$

Since g is strictly increasing, $g'(tf(x)) \neq 0$. Hence,

$$\frac{\partial h(tx)/\partial x_i}{\partial h(tx)/\partial x_j} = \frac{g'(tf(x))t\frac{\partial f(x)}{\partial x_i}}{g'(tf(x))t\frac{\partial f(x)}{\partial x_j}} = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j}.$$

However, the right side of the equation does not depend on $t > 0$ and therefore it is constant in $t > 0$.

What does it say about the level sets? As explained in class, it means that the slopes of the level sets are constant along any ray that origins in 0 (a ray through point x could be parametrized by $t > 0$; i.e., $\text{ray}(x) \equiv \{tx \mid t > 0\}$.

2.12 Let f be a concave function and M the set of global maximizer, i.e.,

$M \equiv \{x^* \in D \mid \text{for all } x \in D,\ f(x^*) \geq f(x)\}$. Obviously, for all $x^1, x^2 \in M$, $m^* \equiv f(x^1) = f(x^2)$.

Let $x^1, x^2 \in M$ and $\lambda \in [0, 1]$ such that $x^\lambda \equiv \lambda x^1 + (1 - \lambda)x^2$.

Since f is concave, $f(x^\lambda) \geq \lambda f(x^1) + (1 - \lambda) f(x^2) = m^*$. Hence,

for all $x \in D$, $f(x^\lambda) \geq m^* \geq f(x)$. Thus, by the definition of M, $x^\lambda \in M$.

2.16

(d) $f(x_1, x_2) = 4x_1 + 2x_2 - x_1^2 + x_1 x_2 - x_2^2$.

Candidate(s) for extrema: $\left\{\frac{28}{3}\right\}$ at $\left\{\left\{x_2 = \frac{8}{3}, x_1 = \frac{10}{3}\right\}\right\}$.

(e) $f(x_1, x_2) = x_1^3 - 6x_1 x_2 + x_2^3$.

Candidate(s) for extrema: $\{0, -8\}$, at $\{x_2 = 0, x_1 = 0\}, \{x_1 = 2, x_2 = 2\}$.