5. Homework, Econ 973,
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5. Homework, Econ 973

3.1 Recall that $\mu_i(x) = \frac{\partial f(x)}{\partial x_i}\frac{x_i}{f(x)}$ and $AP_i(x) = \frac{f(x)}{x_i}$. The elasticity of average product is defined by

$$\begin{aligned}\frac{\partial AP_i(x)}{\partial x_i}\frac{x_i}{AP_i(x)} &= \frac{\partial\left(\frac{f(x)}{x_i}\right)}{\partial x_i}\frac{x_i^2}{f(x)} \\ &= \frac{x_i\frac{\partial f(x)}{\partial x_i} - f(x)}{x_i^2}\frac{x_i^2}{f(x)} \\ &= \frac{x_i\frac{\partial f(x)}{\partial x_i}}{f(x)} - 1 \\ &= \mu_i(x) - 1.\end{aligned}$$

Note that $\frac{\partial AP_i(x)}{\partial x_i} = \frac{x_i\frac{\partial f(x)}{\partial x_i} - f(x)}{x_i^2} = \frac{1}{x_i}[\frac{\partial f(x)}{\partial x_i} - \frac{f(x)}{x_i}] = \frac{1}{x_i}[MP_i(x) - AP_i(x)]$. Hence,

$$\frac{\partial AP_i(x)}{\partial x_i} = \underbrace{\frac{1}{x_i}}_{>0}[MP_i(x) - AP_i(x)]\begin{cases} > 0 & \text{if } MP_i(x) > AP_i(x), \\ = & \text{if } MP_i(x) = AP_i(x), \\ < & \text{if } MP_i(x) < AP_i(x).\end{cases}$$

3.9 Let $f(x) = y = Ax_1^\alpha x_2^\beta$ where $A, \alpha, \beta > 0$. Let $(x_1, x_2) \gg 0$ (hence, divisions by x_1 or x_2 are ok). Then, $f_1(x) = \alpha A x_1^{\alpha-1}x_2^\beta$ and $f_2(x) = \beta A x_1^\alpha x_2^{\beta-1}$. Thus,

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha A x_1^{\alpha-1}x_2^\beta}{\beta A x_1^\alpha x_2^{\beta-1}} = \frac{\alpha}{\beta}\frac{x_2}{x_1}$$

and

$$\begin{aligned} d\ln\frac{f_1(x)}{f_2(x)} &= d\ln\left(\frac{\alpha}{\beta}\frac{x_2}{x_1}\right) \\ &= d\left[\ln\frac{\alpha}{\beta} + \ln\frac{x_2}{x_1}\right] \\ &= d\ln\frac{x_2}{x_1}.\end{aligned}$$

Hence,

$$\begin{aligned}\sigma_{12} &= \frac{d\ln\frac{x_2}{x_1}}{d\ln\frac{f_1(x)}{f_2(x)}} \\ &= 1.\end{aligned}$$

Similarly, $\sigma_{21} = 1$.

3.12 Let $f(x) = y = \frac{k}{1+x_1^{-\alpha}x_2^{-\beta}}$, where $\alpha, \beta > 0$ and $k > y \geq 0$. Then, $f_1(x) = \frac{-(-\alpha x_1^{-\alpha-1}x_2^{-\beta})}{(1+x_1^{-\alpha}x_2^{-\beta})^2}k$ and $f_2(x) = \frac{-(-\beta x_1^{-\alpha}x_2^{-\beta-1})}{(1+x_1^{-\alpha}x_2^{-\beta})^2}k$. Thus,

$$
\begin{aligned}
\frac{f_1(x)}{f_2(x)} &= \frac{\alpha x_1^{-\alpha-1}x_2^{-\beta}}{(1+x_1^{-\alpha}x_2^{-\beta})^2}\frac{(1+x_1^{-\alpha}x_2^{-\beta})^2}{\beta x_1^{-\alpha}x_2^{-\beta-1}} \\
&= \frac{\alpha x_1^{-\alpha-1}x_2^{-\beta}}{\beta x_1^{-\alpha}x_2^{-\beta-1}} = \frac{\alpha x_1^{\alpha}x_2^{\beta+1}}{\beta x_1^{\alpha+1}x_2^{\beta}} \\
&= \frac{\alpha}{\beta}\frac{x_2}{x_1}
\end{aligned}
$$

and

$$
\begin{aligned}
d\ln\frac{f_1(x)}{f_2(x)} &= d\ln\left(\frac{\alpha}{\beta}\frac{x_2}{x_1}\right) \\
&= d\left[\ln\frac{\alpha}{\beta} + \ln\frac{x_2}{x_1}\right] \\
&= d\ln\frac{x_2}{x_1}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma_{12} &= \frac{d\ln\frac{x_2}{x_1}}{d\ln\frac{f_1(x)}{f_2(x)}} \\
&= 1.
\end{aligned}
$$

3.19 Let $c(w, y) = Aw_1^{\alpha}w_2^{\beta}y$.

Since a cost function should be strictly increasing in y, it follows that $A > 0$.

Since a cost function should be increasing in w, it follows that $\alpha, \beta \geq 0$.

Since a cost function should be homogeneous of degree one in w, it follows that $\alpha + \beta = 1$.

$\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ implies that $\alpha, \beta \leq 1$, which guarantees concavity in w.

3.24 Let $y = \min\{\alpha x_1, \beta x_2\}$ where $\alpha, \beta > 0$. It is easy to see that the cheapest way to produce output y is where

$$y = \alpha x_1 = \beta x_2.$$

Hence, $x_1(w,y) = \frac{y}{\alpha}$, $x_2(w,y) = \frac{y}{\beta}$, and $c(w,y) = (\frac{w_1}{\alpha} + \frac{w_2}{\beta})y$.

3.31 Show that for any cost function

$$s_i \equiv \frac{w_i x_i(w,y)}{c(w,y)} = \frac{\partial \ln[c(w,y)]}{\partial \ln w_i}.$$

Calculate

$$\begin{aligned} \frac{\partial \ln[c(w,y)]}{\partial \ln w_i} &= \frac{\partial \ln[c(w,y)]}{\partial w_i} \bigg\backslash \frac{\partial \ln w_i}{\partial w_i} \\ &= \frac{1}{c(w,y)} \frac{\partial c(w,y)}{\partial w_i} \bigg\backslash \frac{1}{w_i} \\ &= \frac{w_i}{c(w,y)} \frac{\partial c(w,y)}{\partial w_i}. \end{aligned}$$

Note that by Shephard's Lemma: $\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y)$. Hence,

$$\frac{\partial \ln[c(w,y)]}{\partial \ln w_i} = \frac{w_i}{c(w,y)} x_i(w,y) = s_i.$$

Check for the Cobb-Douglas cost function $c(w,y) = A \prod_{j=1}^{n} w_j^{\alpha_j} y$:

By Shephard's Lemma: $x_i(w,y) = \frac{\partial c(w,y)}{\partial w_i} = A\alpha_i w_i^{\alpha_i - 1} \prod_{j \neq i} w_j^{\alpha_j} y = A\alpha_i \frac{1}{w_i} \prod_{j=1}^{n} w_j^{\alpha_j} y$.

The input share for input i equals

$$\begin{aligned} s_i &= \frac{w_i x_i(w,y)}{c(w,y)} \\ &= \frac{w_i A \alpha_i \frac{1}{w_i} \prod_{j=1}^{n} w_j^{\alpha_j} y}{A \prod_{j=1}^{n} w_j^{\alpha_j} y} \\ &= \alpha_i. \end{aligned}$$

Next calculate $\frac{\partial \ln[c(w,y)]}{\partial \ln w_i} = \frac{\partial \ln[A \prod_{j=1}^n w_j^{\alpha_j} y]}{\partial \ln w_i}$:

$$\begin{aligned}
\frac{\partial \ln[A \prod_{j=1}^n w_j^{\alpha_j} y]}{\partial \ln w_i} &= \frac{\partial[\ln A + \sum_{j=1}^n \alpha_j \ln w_j + \ln y]}{\partial \ln w_i} \\
&= \underbrace{\frac{\partial \ln A}{\partial \ln w_i}}_{=0} + \frac{\partial(\alpha_i \ln w_i)}{\partial \ln w_i} + \underbrace{\sum_{j \neq i} \frac{\partial(\alpha_j \ln w_j)}{\partial \ln w_i}}_{=0} + \underbrace{\frac{\partial \ln y}{\partial \ln w_i}}_{=0} \\
&= \alpha_i \frac{\partial \ln w_i}{\partial \ln w_i} = \alpha_i.
\end{aligned}$$

3.32 Let $\ln c(w,y) = \alpha_0 + \sum_{i=1}^n \alpha_i \ln w_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln w_i \ln w_j + \ln y$.

(a) We have to restrict parameters such that $c(w,y)$ is linearly homogeneous in w. Thus, for all $t > 0$, $c(tw,y) = tc(w,y)$. Hence, for the translog form,

$$\ln c(tw,y) = \ln(tc(w,y)) = \ln t + \ln c(w,y).$$

We will use that $\sum_{i=1}^n \gamma_{ij} = 0$. Since $\gamma_{ij} = \gamma_{ji}$, we also can use $\sum_{j=1}^n \gamma_{ij} = \sum_{i=1}^n \gamma_{ij} = 0$.

Using the definition of the translog cost function we have

$$\begin{aligned}
\ln c(tw,y) &= \alpha_0 + \sum_{i=1}^n \alpha_i \ln(tw_i) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln(tw_i) \ln(tw_j) + \ln y \\
&= \alpha_0 + \sum_{i=1}^n \alpha_i [\ln t + \ln w_i] \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} [\ln t + \ln w_i][\ln t + \ln w_j] + \ln y \\
&= \alpha_0 + \ln t \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \ln w_i \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \left[(\ln t)^2 + \ln t \ln w_j + \ln t \ln w_i + \ln w_i \ln w_j\right] + \ln y
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_0 + \ln t \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \ln w_i \\
&\quad + \frac{1}{2}(\ln t)^2 \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} \gamma_{ij}}_{=0} + \frac{1}{2} \ln t \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln w_j \\
&\quad + \frac{1}{2} \ln t \sum_{i=1}^{n} \ln w_i \underbrace{\sum_{j=1}^{n} \gamma_{ij}}_{=0} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln w_i \ln w_j + \ln y \\
&= \alpha_0 + \ln t \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \ln w_i \\
&\quad + \frac{1}{2} \ln t \sum_{j=1}^{n} \ln w_j \underbrace{\sum_{i=1}^{n} \gamma_{ij}}_{=0} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln w_i \ln w_j + \ln y \\
&= \alpha_0 + \ln t \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \ln w_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln w_i \ln w_j + \ln y \\
&= \ln(c(w, y) + \ln t \sum_{i=1}^{n} \alpha_i.
\end{aligned}
$$

Hence, $\ln c(tw, y) = \ln t + \ln c(w, y)$ if and only if $\sum_{i=1}^{n} \alpha_i = 1$.

(b) Cobb-Douglas cost function $c^{CD}(w, y) = A \prod_{i=1}^{n} w_i^{\alpha_i} y$. Thus, $\ln c^{CD}(w, y) = \ln A + \sum_{i=1}^{n} \alpha_i \ln w_i + \ln y$. Hence, if $\alpha_0 = \ln A$, $\sum_{i=1}^{n} \alpha_i = 1$, and for all i, j, $\gamma_{ij} = 0$, then the translog cost function reduces to a Cobb-Douglas form.

(c) Input shares in the translog cost function $s_k = \frac{w_k x_k(w,y)}{c(w,y)}$. Note that by Shephard's Lemma $\frac{\partial c(w,y)}{\partial w_k} = x_k(w,y)$. Furthermore, by the definition of the cost function, $c(w,y) = \exp(\alpha_0 + \sum_{i=1}^n \alpha_i \ln w_i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln w_i \ln w_j + \ln y)$. Thus,

$$
\begin{aligned}
x_k(w,y) &= \exp(\alpha_0 + \sum_{i=1}^n \alpha_i \ln w_i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln w_i \ln w_j + \ln y) \\
&\cdot \frac{\partial}{\partial w_k}\left[\alpha_k \ln w_k + \frac{1}{2}[\sum_{i \neq k} \gamma_{ik} \ln w_i \ln w_k + \sum_{j \neq k} \gamma_{kj} \ln w_j \ln w_k + \gamma_{kk}(\ln w_k)^2]\right] \\
&= \exp(\alpha_0 + \sum_{i=1}^n \alpha_i \ln w_i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln w_i \ln w_j + \ln y) \\
&\cdot \left[\frac{\alpha_k}{w_k} + \frac{1}{2}[\sum_{i \neq k} \gamma_{ik} \frac{\ln w_i}{w_k} + \sum_{j \neq k} \gamma_{kj} \frac{\ln w_j}{w_k} + 2\gamma_{kk}\frac{\ln w_k}{w_k}]\right] \\
&= \exp(\alpha_0 + \sum_{i=1}^n \alpha_i \ln w_i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln w_i \ln w_j + \ln y) \\
&\cdot \left[\frac{\alpha_k}{w_k} + \frac{1}{2}[\sum_{i=1}^n \gamma_{ik} \frac{\ln w_i}{w_k} + \sum_{j=1}^n \gamma_{kj} \frac{\ln w_j}{w_k}]\right] \\
&= \exp(\alpha_0 + \sum_{i=1}^n \alpha_i \ln w_i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln w_i \ln w_j + \ln y) \\
&\cdot \left[\frac{\alpha_k}{w_k} + \frac{1}{2}\frac{1}{w_k}\underbrace{[\sum_{i=1}^n \gamma_{ik} \ln w_i + \sum_{j=1}^n \gamma_{kj} \ln w_j]}_{=2\sum_{i=1}^n \gamma_{ik} \ln w_i}\right] \\
&= \exp(\alpha_0 + \sum_{i=1}^n \alpha_i \ln w_i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \ln w_i \ln w_j + \ln y) \\
&\cdot \frac{1}{w_k}\left[\alpha_k + \sum_{i=1}^n \gamma_{ik} \ln w_i\right].
\end{aligned}
$$

Hence,

$$\begin{aligned}\frac{w_k x_k(w,y)}{c(w,y)} &= w_k \frac{1}{w_k}\left[\alpha_k + \sum_{i=1}^{n} \gamma_{ik} \ln w_i\right] \\ &= \alpha_k + \sum_{i=1}^{n} \gamma_{ik} \ln w_i .\end{aligned}$$

Thus, the input shares in the translog function are linear in the logs of input prices and output.

Alternatively, use the result from problem 3.31:

$$\begin{aligned} s_k &= \frac{\partial \ln[c(w,y)]}{\partial \ln w_k} \\ &= \frac{\partial}{\partial \ln w_k}(\alpha_0 + \sum_{i=1}^{n} \alpha_i \ln w_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \ln w_i \ln w_j + \ln y) \\ &= \frac{\partial}{\partial \ln w_k}(\alpha_k \ln w_k + \frac{1}{2}\left[\sum_{i \neq k} \gamma_{ik} \ln w_i \ln w_k + \sum_{j \neq k} \gamma_{kj} \ln w_k \ln w_j + \gamma_{kk}(\ln w_k)^2\right]) \\ &= \alpha_k + \frac{1}{2}\left[\sum_{i \neq k} \gamma_{ik} \ln w_i + \sum_{j \neq k} \gamma_{kj} \ln w_j + 2\gamma_{kk} \ln w_k\right] \\ &= \alpha_k + \sum_{i=1}^{n} \gamma_{ik} \ln w_i .\end{aligned}$$

3.33 Let $y = \sum_{i=1}^{n} \alpha_i x_i$. As usual, $c(w,y) = \min_{x \geq 0} wx$ such that $\sum_{i=1}^{n} \alpha_i x_i \geq y$ (and $\sum_{i=1}^{n} \alpha_i x_i = y$ by strict monotonicity of the production function). Note that inputs are perfect substitutes. Hence, y can be produced by using exactly one of the inputs exclusively. If only input i is used, then we need x_i such that $\alpha_i x_i = y$. The corresponding cost would be $w_i x_i = \frac{w_i}{\alpha_i} y$. Hence, in order to minimize costs, it is sufficient to find an input j such that $\frac{w_j}{\alpha_j} y = \min\{\frac{w_1}{\alpha_1} y, \ldots, \frac{w_n}{\alpha_n} y\} = \min\{\frac{w_1}{\alpha_1}, \ldots, \frac{w_n}{\alpha_n}\} y$. So,

$$c(w,y) = \min\{\frac{w_1}{\alpha_1}, \ldots, \frac{w_n}{\alpha_n}\} y.$$

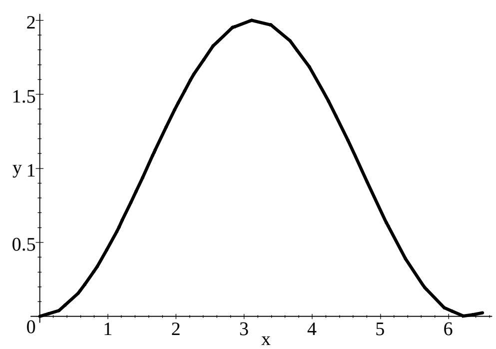
Let $J = \{j \mid \frac{w_j}{\alpha_j} = \min\{\frac{w_1}{\alpha_1}, \ldots, \frac{w_n}{\alpha_n}\}\}$. Then, for all $i = 1, \ldots, n$:

$$x_i(w, y) = \begin{cases} 0 & \text{if } i \notin J, \\ \frac{y}{\alpha_i} & \text{if } J = \{i\}, \\ \lambda_i \frac{y}{\alpha_i} & \text{if } J \supsetneq \{i\} \text{ and } \sum_{j \in J} \lambda_j \frac{y}{\alpha_j} = y \\ & \text{for } \lambda_j \geq 0,\ \sum_{j \in J} \lambda_j = 1. \end{cases}$$

3.46 Let $y = f(x_1, x_2) = x_2 \left[\sin(\frac{x_1}{x_2} - \frac{\pi}{2}) + 1\right]$.

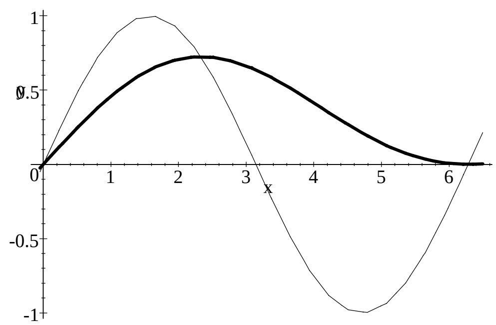
(a) Let $t > 1$. Then, $f(tx_1, tx_2) = tx_2 \left[\sin(\frac{tx_1}{tx_2} - \frac{\pi}{2}) + 1\right] = tx_2 \left[\sin(\frac{x_1}{x_2} - \frac{\pi}{2}) + 1\right] = tf(x_1, x_2)$. Hence, the production function exhibits (global) increasing returns to scale.

(b) $y = f(x_1, 1) = f(x) = \left[\sin(x - \frac{\pi}{2}) + 1\right] = -\cos x + 1$.



The production function $f(x_1, 1) = f(x) = -\cos x + 1$.

Then, $\frac{f(x_1,1)}{x_1} = AP(x) = \frac{-\cos x+1}{x}$ and $\frac{\partial f(x_1,1)}{\partial x_1} = MP(x) = \sin x$.



Marginal (thin line) and Average (thick line) product curves for x_1 when $x_2 = 1$.

(c) $sc(w,y) = \min_{x\geq 0} wx$ such that $f(x) \geq y$. With the given constraints this implies: $sc(y) = sc((1,2),y) = \min_{x_1\geq 0} x_1 + 2 = 2 + \min_{x_1 \geq 0} x_1$ s.t. $y = -\cos x_1 + 1$. Using the specific form of the constraint, we find that the cost minimizer $x_1 = \arccos(1-y) \in [0,\pi]$ and

$$sc(y) = 2 + \arccos(1-y).$$

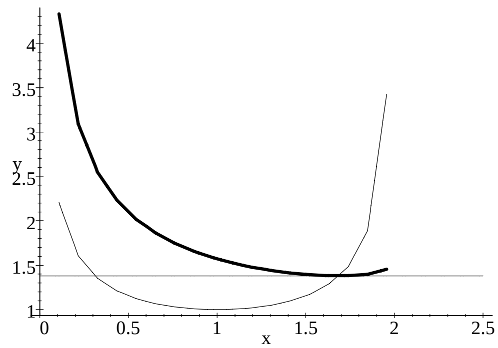
Graphically, you choose the smaller of the two x_1 values that will produce $y = 1 - \cos x_1 = 1 - \cos(2\pi - x_1)$.

Then, $\Pi(p) = \max_{y\geq 0} py - sc(y) = \max_{y\geq 0} py - 2 - \arccos(1-y)$.

FOC (well-definedness analyzed later) $\Rightarrow$

$$\begin{aligned}
p - \frac{-1}{\sqrt{1-(1-y)^2}}(-1) &= 0 \Rightarrow \\
p &= \frac{1}{\sqrt{1-(1-y)^2}} = \frac{1}{\sqrt{2y-y^2}} \\
&\Rightarrow \sqrt{2y-y^2} = \frac{1}{p} \\
&\Rightarrow 2y - y^2 = \frac{1}{p^2} \\
&\Rightarrow y^2 - 2y + \frac{1}{p^2} = 0 \\
&\Rightarrow y_{1\backslash 2} = 1 \pm \sqrt{1 - \frac{1}{p^2}}.
\end{aligned}$$

As we can see, the critical points (and therefore the resulting profit function) are not well-defined for all prices p. Checking the shut-down condition graphically we see that in order to produce a positive output y we need that $p \geq \min savc(y)$ (otherwise profit will be below $-$ fixed costs).



Marginal cost curve (thin line) and average variable cost curve (thick line).

Thus,

$$\begin{aligned} smc(y) &= savc(y) \text{ at the minimum of } savc! \\ \frac{1}{\sqrt{1-(1-y)^2}} &= \frac{\arccos(1-y)}{y} \end{aligned}$$

$\frac{1}{\sqrt{2y-y^2)}} = \frac{\arccos(1-y)}{y} \Leftrightarrow \sqrt{2y-y^2)}\frac{\arccos(1-y)}{y} = 1 \Leftrightarrow \sqrt{2y-y^2)}\frac{\arccos(1-y)}{y} - 1 = 0.$

Using Maple: $\sqrt{2y-y^2)}\frac{\arccos(1-y)}{y} - 1 = 0$, Solution is : $\{y \approx 1.689\,2\}$.

Hence, in order to not shut-down, $p \geq \frac{1}{\sqrt{1-(1-y^*)^2}} = \frac{1}{\sqrt{1-(1-1.689\,2)^2}} = 1.380\,1$.

Then, for $p < 1.380\,1$, $\Pi(p) = -2$ (firm shuts down).

Finally, for $p \geq 1.380\,1$, $\Pi(p) = p + \sqrt{p^2 - 1} - 2 - \arccos\left(-\frac{\sqrt{p^2-1}}{p}\right)$, the profit maximizing output $y^* = 1 + \sqrt{1 - \frac{1}{p^2}} \geq 1.689\,2$ and

$$\begin{aligned}\Pi(p) &= py^* - sc(y^*) \\ &= p + p\sqrt{1 - \frac{1}{p^2}} - 2 - \arccos\left(1 - (1 + \sqrt{1 - \frac{1}{p^2}})\right) \\ &= p + \sqrt{p^2 - 1} - 2 - \arccos\left(-\frac{\sqrt{p^2 - 1}}{p}\right) \geq -2.\end{aligned}$$

3.53 The profit function of the utility equals

$\Pi(p, w_k, w_f) = \max_{K,F \geq 0}(py - w_k K - w_f F)$ s.t. $\sqrt{KF_d} + \sqrt{KF_n} \geq y$, where $y = y_d + y_n$ and $F = F_d + F_n$.

Note that in difference to the standard situation we discussed so far y is constant and $\Pi(p, w_k, w_f) < 0$ is very well possible [since the utility has the obligation to meet all demands at the fixed price p]. $\Rightarrow py$ is a constant.

Furthermore, since the production function is strictly increasing, we may assume that $\sqrt{KF_d} + \sqrt{KF_n} = y$. First, let's minimize costs (necessary condition for profit maximization): $\min_{K,F \geq 0} w_k K + w_f F$ where $\sqrt{KF_d} = y_d$ ($F_d = \frac{y_d^2}{K}$) and $\sqrt{KF_n} = y_n$ ($F_n = \frac{y_n^2}{K}$). Thus, $F = F_d + F_n = \frac{y_d^2 + y_n^2}{K}$ and we have to solve: $\min_{K \geq 0} w_k K + w_f \frac{y_d^2 + y_n^2}{K}$.

FOC: $w_k - w_f \frac{y_d^2 + y_n^2}{K^2} = 0 \Rightarrow \frac{y_d^2 + y_n^2}{K^2} = \frac{w_k}{w_f} \Rightarrow K^* = \sqrt{(y_d^2 + y_n^2)\frac{w_f}{w_k}}$.

Hence, for $y = y_d + y_n = 4 + 3 = 7$,

$$K^* = 5\sqrt{\frac{w_f}{w_k}}.$$