**1. Homework,** Econ973,
Department of Economics
Fall 2000
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## 1. Homework Solutions

**1. Homework Assignment** Problems A1.3 (A1.4), A1.11, A1.12, A1.16 (b), A1.32 (a)&(b), A1.39, A1.46, A1.48.

**1.3** De Morgan's laws:

$$(S \cap T)^c = S^c \cup T^c \tag{1}$$
$$(S \cup T)^c = S^c \cap T^c \tag{2}$$

We prove De Morgan's law (1) by showing that $x \in (S \cap T)^c$ if and only if $x \in S^c \cup T^c$.

$$\begin{aligned} x &\in (S \cap T)^c \\ &\Leftrightarrow x \notin S \cap T \\ &\Leftrightarrow [x \notin S \text{ or } x \notin T] \\ &\Leftrightarrow [x \in S^c \text{ or } x \in T^c] \\ &\Leftrightarrow x \in S^c \cup T^c. \end{aligned}$$

De Morgan's law (2) can be proven similarly by showing that $x \in (S \cup T)^c$ if and only if $x \in S^c \cap T^c$.

$$\begin{aligned} x &\in (S \cup T)^c \\ &\Leftrightarrow x \notin S \cup T \\ &\Leftrightarrow [x \notin S \text{ and } x \notin T] \\ &\Leftrightarrow [x \in S^c \text{ and } x \in T^c] \\ &\Leftrightarrow x \in S^c \cap T^c. \end{aligned}$$

**1.4** Let $I$ be an index set. Then, De Morgan's laws are:

$$(\cap_{i\in I} S_i)^c = \cup_{i\in I} S_i^c \tag{3}$$
$$(\cup_{i\in I} S_i)^c = \cap_{i\in I} S_i^c. \tag{4}$$

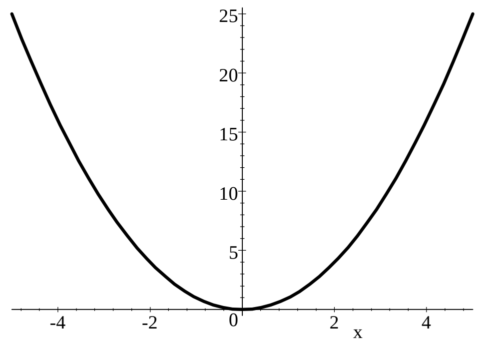
We prove De Morgan's law (3) by showing that $x \in (\cap_{i\in I} S_i)^c$ if and only if $x \in \cup_{i\in I} S_i^c$.

$$\begin{aligned} x &\in (\cap_{i\in I} S_i)^c \\ &\Leftrightarrow x \notin \cap_{i\in I} S_i \\ &\Leftrightarrow [x \notin S_i \text{ for some } i \in I] \\ &\Leftrightarrow [x \in S_i^c \text{ for some } i \in I] \\ &\Leftrightarrow x \in \cup_{i\in I} S_i^c. \end{aligned}$$

De Morgan's law (4) can be proven similarly by showing that $x \in (\cup_{i\in I} S_i)^c$ if and only if $x \in \cap_{i\in I} S_i^c$.

$$\begin{aligned} x &\in (\cup_{i\in I} S_i)^c \\ &\Leftrightarrow x \notin \cup_{i\in I} S_i \\ &\Leftrightarrow [x \notin S_i \text{ for all } i \in I] \\ &\Leftrightarrow [x \in S_i^c \text{ for all } i \in I] \\ &\Leftrightarrow x \in \cap_{i\in I} S_i^c. \end{aligned}$$

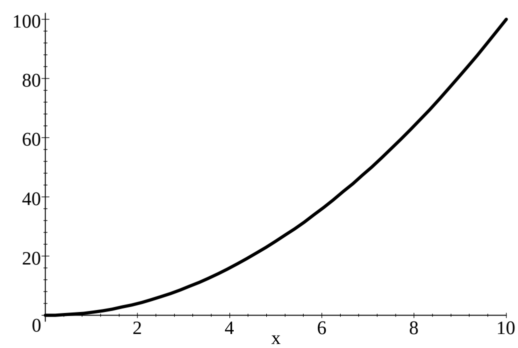
**1.11** Consider $f(x) = x^2$.




Graph of $f(x) = x^2$ where $D = \mathbb{R}$ and $R = \mathbb{R}$ or $R = \mathbb{R}_+$.

**(a)** $D = \mathbb{R}$ and $R = \mathbb{R}$. Then, $I = \mathbb{R}_+$ and the function is neither one-to-one, nor onto.

**(b)** $D = \mathbb{R}$ and $R = \mathbb{R}_+$. Then, $I = \mathbb{R}_+$ and the function is onto, but not one-to-one.




Graph of $f(x) = x^2$ where $D = \mathbb{R}_+$ and $R = \mathbb{R}$ or $R = \mathbb{R}_+$.

**(c)** $D = \mathbb{R}_+$ and $R = \mathbb{R}$. Then, $I = \mathbb{R}_+$ and the function is one-to-one, but not onto.

**(d)** $D = \mathbb{R}_+$ and $R = \mathbb{R}_+$. Then, $I = \mathbb{R}_+$ and the function is one-to-one and onto.

**1.12** The function depicted in Figure A1.8(a) is onto, but it is not one-to-one. Therefore an inverse function does not exist. The function depicted in Figure A1.8(b) is one-to-one, but not onto since $f(D) = f([0,1]) = [0, \frac{1}{2}] \neq [0,1] = R$. For $R = [0, \frac{1}{2}]$ an inverse function would exist.

**1.16 (b)** Let $S$ and $T$ be convex sets. Show that $S - T \equiv \{x \in \mathbb{R}^n \mid x = s - t$ for some $s \in S$ and some $t \in T\}$. Let $\bar{y}, \tilde{y} \in S - T$ and $\lambda \in [0,1]$. To show: $\lambda\bar{y} + (1-\lambda)\tilde{y} \in S - T$ or equivalently:

$$\lambda\bar{y} + (1-\lambda)\tilde{y} = s - t \text{ for some } s \in S \text{ and some } t \in T. \tag{5}$$

Since $\bar{y} \in S - T$, $\bar{y} = \bar{s} - \bar{t}$ for some $\bar{s} \in S$ and some $\bar{t} \in T$.

Since $\tilde{y} \in S - T$, $\tilde{y} = \tilde{s} - \tilde{t}$ for some $\tilde{s} \in S$ and some $\tilde{t} \in T$.

$S$ convex implies $s^\lambda \equiv \lambda\bar{s} + (1-\lambda)\tilde{s} \in S$ and $T$ convex implies $t^\lambda \equiv \lambda\bar{t} + (1-\lambda)\tilde{t} \in T$.

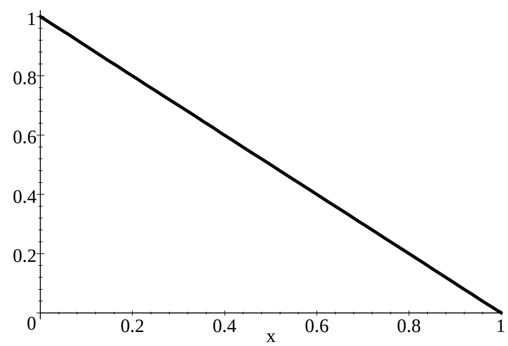
All the above implies that

$$\begin{aligned} \lambda\bar{y} + (1-\lambda)\tilde{y} &= \lambda(\bar{s} - \bar{t}) + (1-\lambda)(\tilde{s} - \tilde{t}) \\ &= [\lambda\bar{s} + (1-\lambda)\tilde{s}] - [\lambda\bar{t} + (1-\lambda)\tilde{t}] \\ &= s^\lambda - t^\lambda, \end{aligned}$$

which proves (5).

**1.32** The $(n-1)$-dimensional unit simplex is defined by $S^{n-1} \equiv \{x \in \mathbb{R}^n \mid x \in \mathbb{R}^n_+$ and $\sum_{i=1}^n x_i = 1\}$.

**(a)** For $n = 2$ we have $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0$ and $x + y = 1\} = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1]$ and $y = 1 - x\}$.




$S^1$ is the line segment connecting the points $(0, 1)$ and $(1, 0)$.

**(b)** In order to prove that $S^{n-1}$ is a convex set, we have to show that for any $x = (x_1, \ldots, x_{n)}) \in S^{n-1}$, any $y = (y_1, \ldots, y_{n)}) \in S^{n-1}$, and any $\lambda \in [0, 1]$:

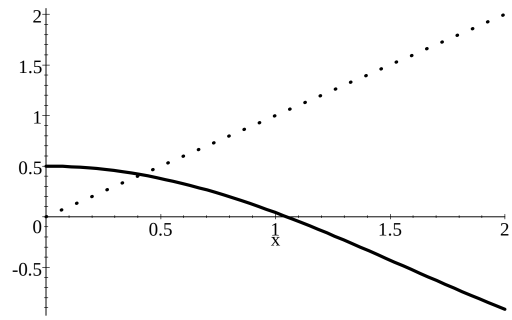
$$\lambda x + (1 - \lambda)y = (\lambda x_1 + (1 - \lambda)y_1, \ldots, \lambda x_n + (1 - \lambda)y_n) \in S^{n-1}.$$

Note that $x \in S^{n-1}$ and $y \in S^{n-1}$ imply that $x, y \in \mathbb{R}_+$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 1$. Since $\lambda \geq 0$ and $x, y \in \mathbb{R}_+$, for all $i \in \{1, \ldots, n\}$: $\lambda x_i + (1 - \lambda)y_i \geq 0$. Hence, $\lambda x + (1 - \lambda)y \in \mathbb{R}^n_+$. Furthermore,

$$\begin{aligned}
\sum_{i=1}^n [\lambda x_i + (1 - \lambda)y_i] &= \sum_{i=1}^n \lambda x_i + \sum_{i=1}^n (1 - \lambda)y_i \\
&= \lambda \underbrace{\sum_{i=1}^n x_i}_{=1} + (1 - \lambda) \underbrace{\sum_{i=1}^n y_i}_{=1} \\
&= \lambda + (1 - \lambda) \\
&= 1.
\end{aligned}$$

Hence, $\sum_{i=1}^n [\lambda x_i + (1 - \lambda)y_i] = 1$ and $\lambda x + (1 - \lambda)y \in \mathbb{R}^n_+$. This proves that $\lambda x + (1 - \lambda)y \in S^{n-1}$.

**1.39** Use Brouwer's fixed point theorem to show that the equation $\cos(x) - x - \frac{1}{2} = 0$ has a solution in the interval $[0, \frac{\pi}{4}]$. Let $f(x) = \cos(x) - \frac{1}{2}$.




The functions $f(x) = \cos x - \frac{1}{2}$ and $\tilde{f}(x) = x$.

We prove that $f(x)$ has a fixed point in the interval $[0, \frac{\pi}{4}]$. Note that on $[0, \frac{\pi}{4}]$, $f(x)$ is a strictly decreasing. Thus,

$$f([0, \frac{\pi}{4}]) = [f(\frac{\pi}{4}), f(0)] = [\frac{1}{2}\sqrt{2} - \frac{1}{2}, \frac{1}{2}] \subsetneq [0.2, 0.5] \subsetneq [0, \frac{\pi}{4}].$$

Let $S \equiv [0, \frac{\pi}{4}]$. Hence, $S$ is nonempty, convex, and compact (closed and bounded). Furthermore, $f : S \to S$ is continuous (cos is a continuous function, so $\cos x - \frac{1}{2}$ is also continuous). Hence, using Brouwer's fixed point theorem, there exists a fixed point $x^* \in S$ such that $f(x^*) = x^*$. Thus, for $x^* \in [0, \frac{\pi}{4}]$, $\cos(x^*) - \frac{1}{2} = x^* \Leftrightarrow \cos(x^*) - x^* - \frac{1}{2} = 0$. So, $x^*$ solves the equation $\cos(x) - x - \frac{1}{2} = 0$ in the interval $[0, \frac{\pi}{4}]$.

**1.46** Let $f(x) = \mathbf{a} \cdot \mathbf{x} + b$ where $\mathbf{a}, \mathbf{x} \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

**(a)** $f$ being convex and concave means that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\lambda \in [0, 1]$

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \text{ (convexity)}$$

and

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \text{ (concavity).}$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. We have

$$\begin{aligned} f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) &= \mathbf{a} \cdot [\lambda \mathbf{x} + (1-\lambda)\mathbf{y}] + b \\ &= \lambda \mathbf{a} \cdot \mathbf{x} + (1-\lambda)\mathbf{a} \cdot \mathbf{y} + \lambda b + (1-\lambda) b \\ &= \lambda(\mathbf{a} \cdot \mathbf{x} + b) + (1-\lambda)(\mathbf{a} \cdot \mathbf{y} + b) \\ &= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}). \end{aligned}$$

Hence, $f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$, which proves that $f$ is convex and concave. However, if $f$ is strictly convex and strictly concave, then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \mathbf{y}$, and all $\lambda \in (0, 1)$

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \text{ (strict convexity)}$$

and

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) > \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \text{ (strict concavity).}$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \mathbf{y}$, and $\lambda \in (0, 1)$. Then, similarly as before it follows that $f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$. This shows that $f$ is neither strictly convex, nor strictly concave.

**(b)** Since a linear function $f$ is convex, it must be quasiconvex. Similarly, since a linear function $f$ is concave, it must be quasiconcave. If $f$ is strictly quasiconvex and strictly quasiconcave, then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \mathbf{y}$, and all $\lambda \in (0, 1)$

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) < \max\{f(\mathbf{x}), f(\mathbf{y})\} \text{ (strict quasiconvexity)}$$

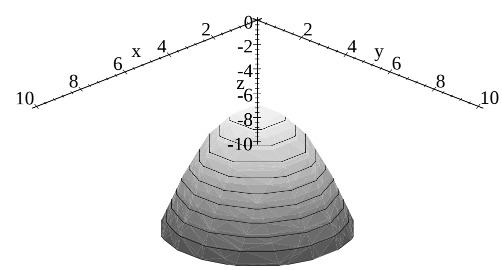
and

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) > \min\{f(\mathbf{x}), f(\mathbf{y})\} \text{ (strict quasiconcavity).}$$

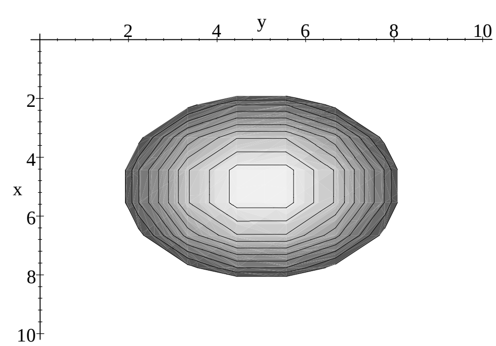
Recall that $f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) = \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$. So, if $\min\{f(\mathbf{x}), f(\mathbf{y})\} < \max\{f(\mathbf{x}), f(\mathbf{y})\}$, then the conditions for strict quasiconvexity and strict quasiconcavity are satisfies. But for any $\mathbf{a} \in \mathbb{R}^n$ we can find $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \mathbf{y}$ and $f(\mathbf{x}) = f(\mathbf{y})$ ($\Leftrightarrow \mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{y}$). Therefore, $f$ is neither strictly quasiconvex, nor strictly quasiconcave.

**1.48** There are several ways of proving that $f(x_1, x_2) = -(x_1 - 5)^2 - (x_2 - 5)^2$ is quasiconcave. The easiest way probably is by using that a function is quasiconcave iff the superior sets for all levels $y$, $S(y)$, are convex.

$$\begin{aligned}
S(y) &= \emptyset \text{ for levels } y > 0 \text{ is a convex set,} \\
S(y) &= \{(5,5)\} \text{ for level } y = 0 \text{ is a convex set,} \\
S(y) &= \{(x_1, x_2) \mid -(x_1 - 5)^2 - (x_2 - 5)^2 \geq y\} \\
&= \{(x_1, x_2) \mid (x_1 - 5)^2 + (x_2 - 5)^2 \leq -y\} \neq \emptyset \text{ for levels } y < 0 \\
&= B^*_{\sqrt{-y}}((5,5)) \text{ is a convex set.}
\end{aligned}$$




The function $z = -(x - 5)^2 - (y - 5)^2$.




The function $z = -(x - 5)^2 - (y - 5)^2$: superior sets.

Alternatively, we can show that $f(x_1, x_2)$ is a strictly concave function using the definition of strict concavity. Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, $\bar{x} \neq \tilde{x}$, and $\lambda \in (0, 1)$. Then,

$$f(\lambda\bar{x} + (1-\lambda)\tilde{x})$$

$$= -[\lambda\bar{x}_1 + (1-\lambda)\tilde{x}_1 - 5]^2 - [\lambda\bar{x}_2 + (1-\lambda)\tilde{x}_2 - 5]^2$$

$$= -[\lambda(\bar{x}_1 - 5) + (1-\lambda)(\tilde{x}_1 - 5)]^2 - [\lambda(\bar{x}_2 - 5) + (1-\lambda)(\tilde{x}_2 - 5)]^2$$

$$= -[\lambda^2(\bar{x}_1 - 5)^2 + (1-\lambda)^2(\tilde{x}_1 - 5)^2 + 2\lambda(1-\lambda)(\bar{x}_1 - 5)(\tilde{x}_1 - 5)]$$

$$-[\lambda^2(\bar{x}_2 - 5)^2 + (1-\lambda)^2(\tilde{x}_2 - 5)^2 + 2\lambda(1-\lambda)(\bar{x}_2 - 5)(\tilde{x}_2 - 5)]$$

$$= -[\lambda(\bar{x}_1 - 5)^2(1-(1-\lambda)) + (1-\lambda)(\tilde{x}_1 - 5)^2(1-\lambda) + 2\lambda(1-\lambda)(\bar{x}_1 - 5)(\tilde{x}_1 - 5)]$$

$$-[\lambda(\bar{x}_2 - 5)^2(1-(1-\lambda)) + (1-\lambda)(\tilde{x}_2 - 5)^2(1-\lambda) + 2\lambda(1-\lambda)(\bar{x}_2 - 5)(\tilde{x}_2 - 5)]$$

$$= -[\lambda(\bar{x}_1 - 5)^2 + (1-\lambda)(\tilde{x}_1 - 5)^2]$$

$$+\lambda(1-\lambda)(\bar{x}_1 - 5)^2 + \lambda(1-\lambda)(\tilde{x}_1 - 5)^2 - 2\lambda(1-\lambda)(\bar{x}_1 - 5)(\tilde{x}_1 - 5)$$

$$-[\lambda(\bar{x}_2 - 5)^2 + (1-\lambda)(\tilde{x}_2 - 5)^2]$$

$$+\lambda(1-\lambda)(\bar{x}_2 - 5)^2 + \lambda(1-\lambda)(\tilde{x}_2 - 5)^2 - 2\lambda(1-\lambda)(\bar{x}_2 - 5)(\tilde{x}_2 - 5)$$

$$= \lambda[-(\bar{x}_1 - 5)^2 - (\bar{x}_2 - 5)^2] + (1-\lambda)[-(\tilde{x}_1 - 5)^2 - (\tilde{x}_2 - 5)^2]$$

$$+ \underbrace{\lambda(1-\lambda)}_{>0} \underbrace{[((\bar{x}_2 - 5) - (\tilde{x}_2 - 5))^2 + ((\bar{x}_2 - 5) - (\tilde{x}_2 - 5))^2]}_{>0 \text{ (=0 only for } \bar{x}=\tilde{x})}$$

$$> \lambda[-(\bar{x}_1 - 5)^2 - (\bar{x}_2 - 5)^2] + (1-\lambda)[-(\tilde{x}_1 - 5)^2 - (\tilde{x}_2 - 5)^2]$$

$$= \lambda f(\bar{x}) + (1-\lambda) f(\tilde{x}).$$

Hence, $f(\lambda\bar{x} + (1-\lambda)\tilde{x}) > \lambda f(\bar{x}) + (1-\lambda) f(\tilde{x})$, which proves that $f(x_1, x_2)$ is a strictly concave function. Hence, $f(x_1, x_2)$ is also quasiconcave.