2. Homework, Part II, Econ 973,
Department of Economics
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A2.19 $f(x_1, x_2) = x_1^2 x_2^2$.

Then, $H(x) = \begin{pmatrix} 2x_2^2 & 4x_1x_2 \\ 4x_1x_2 & 2x_1^2 \end{pmatrix}$ and $z^T H(x) z = 2z_1^2 x_2^2 + 8z_1 z_2 x_1 x_2 + 2z_2^2 x_1^2$.

Let $z = (1, 1)$ and $x = (1, 1)$. Then, $z^T H(x) z = 12 > 0$ (locally strictly convex). Therefore, f cannot be concave.

Let $z = (-1, 1)$ and $x = (1, 1)$. Then, $z^T H(x) z = -4 < 0$ (locally strictly concave). Therefore, f cannot be convex.

In order to decide about quasiconcavity we need to look at the superior sets of the function.

$$\begin{aligned} S(y) &= \mathbb{R}^2_+ \text{ for levels } y \leq 0 \text{ is a convex set,} \\ S(y) &= \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 \neq 0, x_2 \neq 0, x_1^2 x_2^2 \geq y\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 \neq 0, x_2 \neq 0, x_2 \geq \frac{\sqrt{y}}{x_1}\} \neq \emptyset. \end{aligned}$$

We need to show that for levels $y > 0$, $S(y)$ is a convex set. Recall that if $y > 0$, then $x_1 \neq 0$ and $x_2 \neq 0$. So, we can define $h(x_1) = \frac{\sqrt{y}}{x_1}$. Then, $S(y) = \text{epi}(h)$. We know that $\text{epi}(h)$ is a convex set if and only if h is a convex function. Note that $h'(x_1) = -\frac{\sqrt{y}}{x_1^2}$ and $h''(x_1) = 2\frac{\sqrt{y}}{x_1^3} > 0$. Hence, h is a strictly convex function. Therefore, $\text{epi}(h) = S(y)$ is a convex set. Hence, $f(x_1, x_2) = x_1^2 x_2^2$ is quasiconcave.

A2.24

(d) $y = 4x_1 + 2x_2 - x_1^2 + x_1 x_2 - x_2^2$.

FOC^s $\Rightarrow$ Critical point $(x_1, x_2) = (\frac{10}{3}, \frac{8}{3})$.

$H(x) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ and $z^T H(x) z = -2z_1^2 + 2z_1 z_2 - 2z_2^2 < 0$ for $z \neq 0$.[1]

So $H(x)$ is negative definite. Hence, f has a local maximum at $(\frac{10}{3}, \frac{8}{3})$ and $f(\frac{10}{3}, \frac{8}{3}) = \frac{28}{3}$.

[1] If $z_1 z_2 \leq 0$ it is clear that $-2z_1^2 + 2z_1 z_2 - 2z_2^2 < 0$. If $z_1 z_2 > 0$, then $-2z_1^2 + 2z_1 z_2 - 2z_2^2 = -z_1^2 - z_2^2 - (z_1 - z_2)^2 < 0$.

(e) $y = x_1^3 - 6x_1x_2 + x_2^3$.

FOC^s $\Rightarrow$ Critical points $(0,0)$ and $(2,2)$..

$H(x) = \begin{pmatrix} 6x_1 & -6 \\ -6 & 6x_2 \end{pmatrix}$ and $z^T H(x) z = 6z_1^2 x_1 - 12z_1z_2 + 6z_2^2 x_2$ (so $H(x)$'s definiteness depends on the values x_1 and x_2).

For $(x_1, x_2) = (0,0)$, $z^T H(0) z = -12z_1z_2$. Let $x_1 = 0$. Then, $f(0, x_2) = x_2^3$ does not have a local maximum or minimum at $(0, x_2) = (0,0)$. Thus, the function $f(x_1, x_2)$ cannot have a local maximum or minimum at $(0, x_2) = (0,0)$ as well.

For $(x_1, x_2) = (2,2)$, $z^T H(x) z = 12z_1^2 - 12z_1z_2 + 12z_2^2 > 0$ for $z \neq 0$, $H(x)$ is positive definite. Hence, f has a local minimum at $(2,2)$ and $f(2,2) = -8$.

A2.25 Use Lagrange to solve the maximization problems (calculations here not included).

(d) $\max_{x_1,x_2} x_1 + x_2$ s.t. $x_1^4 + x_2^4 = 1$. The objective function is continuous and the feasible set is compact. Hence, by the Theorem of Weierstraß, the objective function attains its maximum and minimum on the feasible set. Thus, by evaluating the functional values in all critical points, minimum and maximum can be found.

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 + x_2 + \lambda(x_1^4 + x_2^4 - 1).$$

Necessary FOC^s:

$$\begin{aligned} 4\lambda x_1^3 &= 1 \\ 4\lambda x_2^3 &= 1 \\ x_1^4 + x_2^4 &= 1 \end{aligned}$$

It is obvious that $\lambda \neq 0$. Hence, we can divide the first two equations and obtain $x_1 = x_2$. Plugging this into the third equation yields two critical points of the objective function: $x_1 = x_2 = 2^{-\frac{1}{4}}$ and $x_1 = x_2 = -2^{-\frac{1}{4}}$. It is obvious that the objective function attains its maximum at $(2^{-\frac{1}{4}}, 2^{-\frac{1}{4}})$ and its minimum at $(-2^{-\frac{1}{4}}, -2^{-\frac{1}{4}})$ with corresponding functional values $2^{\frac{3}{4}} \approx 1.68$ and $-2^{\frac{3}{4}} \approx -1.68$.

(e) $\max_{x_1,x_2,x_3} x_1 x_2^2 x_3^3$ s.t. $x_1 + x_2 + x_3 = 1$. Note that the feasible set is a hyperplane; it is not bounded and therefore not compact. With Lagrange and using Second Order Conditions, you can find that $(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ is a local maximizer: the corresponding bordered Hessian equals

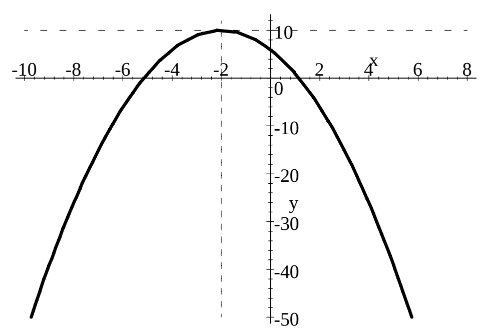
$$\bar{H} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \frac{1}{12} & \frac{1}{12} \\ 1 & \frac{1}{12} & \frac{1}{24} & \frac{1}{12} \\ 1 & \frac{1}{12} & \frac{1}{12} & \frac{1}{18} \end{pmatrix}.$$

Then, $\bar{D}_2 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{1}{12} \\ 1 & \frac{1}{12} & \frac{1}{24} \end{vmatrix} = \frac{1}{8} > 0$, $\bar{D}_3 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \frac{1}{12} & \frac{1}{12} \\ 1 & \frac{1}{12} & \frac{1}{24} & \frac{1}{12} \\ 1 & \frac{1}{12} & \frac{1}{12} & \frac{1}{18} \end{vmatrix} = -\frac{1}{144} < 0.$

This is sufficient to identify $(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ as a local maximizer with function value $\frac{1}{432}$.

However, $(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ is not a global maximizer. To show this, let $\bar{x}_3 = 1$, $\bar{x}_1 = t > 0$ and $\bar{x}_2 = -t$. Then, $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1$ and $\max_{x_1,x_2,x_3} \bar{x}_1 \bar{x}_2^2 \bar{x}_3^3 = t^3 \underset{t\to\infty}{\longrightarrow} \infty$. A maximum of the objective function on the feasible set does not exist!

A2.26 $f(x) = 6 - x^2 - 4x$.



Graph of $f(x) = 6 - x^2 - 4x$. The function attains its global maximum at $x = -2$.

Foc: $f'(x) = -2x - 4 = 0 \Rightarrow x = -2$ (critical point). SOC: $f''(x) = -2 < 0 \Rightarrow f$ is strictly convex $\Rightarrow f$ attains a local and a global maximum at $x = -2$.

Now add the nonnegativity constraint $x \geq 0$. Note that then $f'(x) = -2x - 4 < 0$ Therefore, f will now attain its maximal value $x = 0$: $f(0) = 6 < 10$.

See extra copies for the solutions of A and B.