**3. Homework, Part I,** Econ 973,
Department of Economics
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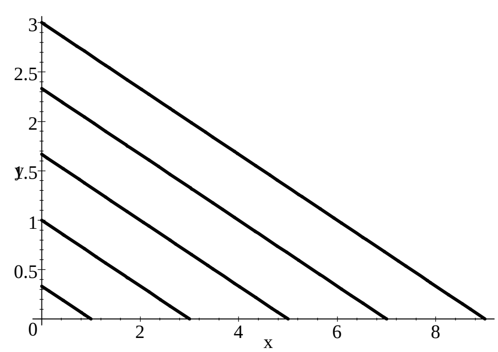
**1.6** An example for non-convex preferences are so-called "single-dipped" (or double-peaked) preferences which are induced by situations where pure states are preferred to mixtures. Single-dippedness means that there is a worst bundle (e.g., you don't like beer mixed with water), but the purer the bundle is, the better it gets (e.g., I like fresh water and pure (German) beer). For public goods economies, an example would be the location of a "bad" (e.g., a waste disposal site - the further away you live from it, the better it is). One-dimensional example:

**1.8** All preferences as described in 1.8 can be represented by a utility function $u(x, y) = \alpha x + \beta y$ such that $\alpha, \beta > 0$.




Indifference curves of $u(x, y) = x + 3y$.

Proof that any $u(x_1, x_2) = \alpha x_1 + \beta x_2$ such that $\alpha, \beta > 0$ represents a convex preference relation $\succeq$, but not a strictly convex preference relation:

In order to prove convexity, we need to show that for any $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $\succeq(\bar{x})$ is a convex set.

Note that $\succeq(\bar{x}) = \{x \in \mathbb{R}^2_+ \mid x \succeq \bar{x}\} = \{x \in \mathbb{R}^2_+ \mid \alpha x_1 + \beta x_2 \geq \alpha\bar{x}_1 + \beta\bar{x}_2\}$.

Let $\tilde{x}, \hat{x} \in \succeq(\bar{x})$ and $\lambda \in [0,1]$. We show that $x^\lambda = \lambda\tilde{x} + (1-\lambda)\hat{x} \in \succeq(\bar{x})$.

$$\begin{aligned}
\tilde{x} &\in \succeq(\bar{x}) \Rightarrow \alpha\tilde{x}_1 + \beta\tilde{x}_2 \geq \alpha\bar{x}_1 + \beta\bar{x}_2 \\
&\Rightarrow \lambda(\alpha\tilde{x}_1 + \beta\tilde{x}_2) \geq \lambda(\alpha\bar{x}_1 + \beta\bar{x}_2), \qquad (1)\\
\hat{x} &\in \succeq(\bar{x}) \Rightarrow \alpha\hat{x}_1 + \beta\hat{x}_2 \geq \alpha\bar{x}_1 + \beta\bar{x}_2 \\
&\Rightarrow (1-\lambda)(\alpha\hat{x}_1 + \beta\hat{x}_2) \geq (1-\lambda)(\alpha\bar{x}_1 + \beta\bar{x}_2). \qquad (2)
\end{aligned}$$

Adding up inequalities (1) and (2) yields

$$\begin{aligned}
\lambda(\alpha\tilde{x}_1 + \beta\tilde{x}_2) + (1-\lambda)(\alpha\hat{x}_1 + \beta\hat{x}_2) &\geq \alpha\bar{x}_1 + \beta\bar{x}_2 \Rightarrow \\
\alpha(\lambda\tilde{x}_1 + (1-\lambda)\hat{x}_1) + \beta(\lambda\tilde{x}_2 + (1-\lambda)\hat{x}_2 &\geq \alpha\bar{x}_1 + \beta\bar{x}_2 \Rightarrow \\
x^\lambda = \lambda\tilde{x} + (1-\lambda)\hat{x} \in & \succeq(\bar{x}).
\end{aligned}$$

Next, we prove that $\succeq$ does not satisfy strict convexity: Let $\bar{x} \neq \tilde{x}$ such that $\alpha\bar{x}_1 + \beta\bar{x}_2 = \alpha\tilde{x}_1 + \beta\tilde{x}_2$. Then, $\bar{x} \succeq \tilde{x}$. Let $\lambda \in (0,1)$. Thus, for $x^\lambda = \lambda\bar{x} + (1-\lambda)\tilde{x}$,

$$\begin{aligned}
\alpha x_1^\lambda + \beta x_2^\lambda &= \alpha(\lambda\bar{x}_1 + (1-\lambda)\tilde{x}_1) + \beta(\lambda\bar{x}_2 + (1-\lambda)\tilde{x}_2) \\
&= \lambda(\alpha\bar{x}_1 + \beta\bar{x}_2) + (1-\lambda)(\alpha\tilde{x}_1 + \beta\tilde{x}_2) \\
&= \alpha\tilde{x}_1 + \beta\tilde{x}_2.
\end{aligned}$$

Hence, $x^\lambda \sim \tilde{x}$. However, for strict convexity we need that $x^\lambda \succ \tilde{x}$.

Thus, $\succeq$ is convex, but not strictly convex.

**1.12 (b)** Prove that if $u(x)$ and $v(x)$ are quasiconcave, then $m(x) \equiv \min\{u(x), v(x)\}$ is quasiconcave.

Let $x^1, x^2 \in \mathbb{R}^2_+$, $\lambda \in [0,1]$, and $x^\lambda \equiv \lambda x^1 + (1-\lambda)x^2$. To show:

$$\begin{aligned}
m(x^\lambda) &\geq \min\{m(x^1), m(x^2)\} \Leftrightarrow \\
\min\{u(x^\lambda), v(x^\lambda)\} &\geq \min\{\min\{u(x^1), v(x^1)\}, \min\{u(x^2), v(x^2)\}\}. \qquad (3)
\end{aligned}$$

Quasiconcavity of $u$ and $v$ implies

$$u(x^\lambda) \geq \min\{u(x^1), u(x^2)\} \text{ and } v(x^\lambda) \geq \min\{v(x^1), v(x^2)\}.$$

Suppose that $\min\{u(x^\lambda), v(x^\lambda)\} = u(x^\lambda)$.

So,

$$\begin{aligned}\min\{u(x^\lambda), v(x^\lambda)\} &= u(x^\lambda) \geq \min\{u(x^1), u(x^2)\} \\ &\geq \min\{u(x^1), v(x^1), u(x^2), v(x^2)\} \\ &= \min\{\min\{u(x^1), v(x^1)\}, \min\{u(x^2), v(x^2)\},\end{aligned}$$

the desired inequality (3).

Suppose that $\min\{u(x^\lambda), v(x^\lambda)\} = v(x^\lambda)$.

So,

$$\begin{aligned}\min\{u(x^\lambda), v(x^\lambda)\} &= v(x^\lambda) \geq \min\{v(x^1), v(x^2)\} \\ &\geq \min\{u(x^1), v(x^1), u(x^2), v(x^2)\} \\ &= \min\{\min\{u(x^1), v(x^1)\}, \min\{u(x^2), v(x^2)\},\end{aligned}$$

the desired inequality (3).

**1.13** **(a)** Note that indifference sets are singletons (i.e., points). So, an indifference map could look like this:

Depicting an upper contour set may make it clearer how lexicographic preferences "look like".

**(b)** As we can see, weak upper contour sets are not closed. Hence, lexicographic preferences are not continuous. Therefore, we cannot represent lexicographic preferences by a utility function.

**1.20** Let $u(x) = Ax_1^{\alpha}x_2^{1-\alpha}$ where $A > 0$ and $\alpha \in (0,1)$. Let $(p, y) \gg 0$.

$$\max_{x \in \mathbb{R}^2_+} u(x) \text{ s.t. } px \leq y.$$

The constraint set $B(p, y) \equiv \{x \in \mathbb{R}^2_+ \mid px \leq y\}$ is compact and nonempty and $u$ is continuous (even $C^1$). By Weierstraß, maximizer and minimizer exist on $B(p, y)$. Furthermore, $u(x) = 0$ if $x \notin \mathbb{R}^2_{++}$ (i.e., if $x_1 = 0$ or $x_2 = 0$). For all $x \in \mathbb{R}^2_{++}$, $u(x) > 0$. Hence, all $x \in \mathbb{R}^2_+ \backslash \mathbb{R}^2_{++}$ are minimizer and any maximizer will be an interior point (i.e., $x \in \mathbb{R}^2_{++}$).

Furthermore, $\frac{\partial u(x)}{\partial x_1} = A\alpha x_1^{\alpha-1}x_2^{1-\alpha} > 0$ and $\frac{\partial u(x)}{\partial x_2} = A(1-\alpha)x_1^{\alpha}x_2^{-\alpha} > 0$. Hence, $u$ is strictly monotonic $u$ and we will have a balanced budget at the maximum (i.e., $px = y$).

Solving Lagrange's FOC[s] will produce candidates for the maximizer

(i.e., the Marshallian demand).

$\mathcal{L}(x, \lambda) \equiv A x_1^{\alpha} x_2^{1-\alpha} + \lambda(y - px).$

$$\begin{aligned} \frac{\partial \mathcal{L}(x,\theta)}{\partial x_1} &= 0 \Rightarrow A\alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1, && (4) \\ \frac{\partial \mathcal{L}(x,\theta)}{\partial x_2} &= 0 \Rightarrow A(1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2, && (5) \\ \frac{\partial \mathcal{L}(x,\theta)}{\partial \lambda} &= 0 \Rightarrow y = px. && (6) \end{aligned}$$

Since $A, \alpha, p_1 > 0$ and $x \in \mathbb{R}^2_{++}$, $\lambda > 0$. Furthermore, all divisions by $x_i, p_i$, and $\lambda$ are permitted. Dividing equation (4) by (5) yields

$$\frac{\alpha x_2}{(1-\alpha)x_1} = \frac{p_1}{p_2}.$$

Hence,

$$x_2 = \frac{(1-\alpha)x_1}{\alpha} \frac{p_1}{p_2}.$$

Substituting into (6), we obtain

$$\begin{aligned} p_1 x_1 + p_1 \frac{(1-\alpha)x_1}{\alpha} &= y, \\ p_1 x_1 &= \alpha y. \end{aligned}$$

Thus,

$$x_1(p, y) = \frac{\alpha y}{p_1} \text{ and } x_2(p, y) = \frac{(1-\alpha)y}{p_2}. \quad (7)$$

By Weierstraß $x(p, y)$ as defined in (7) is the solution to the consumer's utility maximization problem.

**1.24** Recall that $f$ represents the same preferences as $u$ if and only if there exists a strictly monotonic transformation $v : \mathbb{R} \to \mathbb{R}$ ($v$ is strictly increasing on the range of $u$) such that $f = v \circ u$; i.e., for all $x \in \mathbb{R}^n_+$, $f(x) = v(u(x))$. Denote the range of $u$ by $R(u)$.

**(a)** $f(x) = u(x) + (u(x))^3$. Hence, $f = v \circ u$ where for all $y \in R(u)$, $v(y) = y + y^3$. Note that $\frac{\partial v(y)}{\partial y} = 1 + 3y^2 > 0$ for all $y \in \mathbb{R}$. Thus, $v$ is strictly increasing on $R(u)$ and $f$ represents the same preferences as $u$.

**(b)** $f(x) = u(x) - (u(x))^2$. Hence, $f = v \circ u$ where for all $y \in R(u)$, $v(y) = y - y^2$. Note that $\frac{\partial v(y)}{\partial y} = 1 - 2y \leq 0$ for all $y \geq \frac{1}{2}$ (strict-monotonicity is violated in that case). Hence, if $u(x) < \frac{1}{2}$ for all $x \in \mathbb{R}^n_+$, then $f$ represents the same preferences as $u$. If there exists $x \in \mathbb{R}^n_+$ such that $u(x) \geq \frac{1}{2}$, then $f$ cannot represent the same preferences as $u$.

**(c)** $f(x) = u(x) + \sum_{i=1}^n x_i$. In this case, we don't have enough information to find a transformation $v$ such that $f = v \circ u$ (the term $\sum_{i=1}^n x_i$ generally cannot be expressed in terms of $u(x)$). When could we find a transformation such that $f = v \circ u$? Iff for all bundles $\bar{x}, \tilde{x} \in \mathbb{R}^n_+$, $u(\bar{x}) \geq u(\tilde{x}) \Leftrightarrow f(\bar{x}) \geq f(\tilde{x}) \Leftrightarrow u(\bar{x}) + \sum_{i=1}^n \bar{x}_i \geq u(\tilde{x}) + \sum_{i=1}^n \tilde{x}_i$. If $u(x) = h(\sum_{i=1}^n x_i)$ where $h : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing function, then $v = y + h^{-1}(y)$ for all $y \in R(u) = \mathbb{R}_+$ would be a strictly monotonic transformation such that $f = v \circ u$. In this case $f$ would represent the same preferences as $u$. Otherwise, not.