3. Homework, Part II, Econ 973,
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3. Homework, Part II, Econ 973,

1.27 Let $a \in (0,1)$ and $u(x) = \max\{ax_1, ax_2\} + \min\{x_1, x_2\}$.

First, consider the case that $x_1 \geq x_2$. Then, $u(x) = ax_1 + x_2$.

Second, consider the case that $x_1 \leq x_2$. Then, $u(x) = ax_2 + x_1$.

(Obviously, for $x_1 = x_2$, $u(x) = ax_1 + x_2 = ax_2 + x_1$.)

A graph showing the indifference map is included on the next page (not included in the file).

Hence, indifference curves are piecewise linear with slopes $-a$ (if $x_1 > x_2$) and $-\frac{1}{a}$ (if $x_1 < x_2$. Note that for $x_1 = x_2$, the utility function is not differentiable.

Note that $-\frac{1}{a} < -a$ or equivalently $\frac{1}{a} > a$. For the Marshallian demands this implies five different cases which I solved graphically; a graph for each case is included on the next page (not included in the file).

Case 1 $\frac{p_1}{p_2} < a$:

$$x(p,y) = \left(\frac{y}{p_1}, 0\right).$$

Case 2 $\frac{p_1}{p_2} = a$ (multiple solutions denoted as correspondence)

$$\begin{aligned} x(p,y) &= \left\{\alpha\left(\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2}\right) + (1-\alpha)\left(\frac{y}{p_1}, 0\right) |\alpha \in [0,1]\right\} \\ &= \left\{(x_1,x_2) \middle| x_1 \geq x_2 \text{ and } x_2 = \frac{y}{p_2} - \frac{p_1 x_1}{p_2} = \frac{y}{p_2} - ax_1\right\}. \end{aligned}$$

Case 3 $\frac{1}{a} > \frac{p_1}{p_2} > a$:

$$x(p,y) = \left(\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2}\right).$$

Case 4 $\frac{p_1}{p_2} = \frac{1}{a}$ (multiple solutions denoted as correspondence)

$$\begin{aligned} x(p,y) &= \left\{\alpha\left(\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2}\right) + (1-\alpha)\left(0, \frac{y}{p_2}\right) |\alpha \in [0,1]\right\} \\ &= \left\{(x_1,x_2) \middle| x_1 \leq x_2 \text{ and } x_1 = \frac{y}{p_1} - \frac{p_2 x_2}{p_1} = \frac{y}{p_1} - ax_2\right\}. \end{aligned}$$

Case 5 $\frac{p_1}{p_2} > \frac{1}{a}$:

$$x(p,y) = \left(0, \frac{y}{p_2}\right).$$

1.28 Let $\beta \in (0,1)$. In order to calculate the optimal consumption level, we have to solve

$$\max u(x_0, x_1, \ldots) = \max \sum_{t=0}^{\infty} \beta^t \ln(x_t) \text{ s.t. } \sum_{t=0}^{\infty} x_t = 1.$$

Note that $\ln(0) = -\infty$. Hence, at the maximum, we must have $x_t > 0$ for all periods.

To make our life easy, we start with an agent that lives k periods (later $k \to \infty$). So, first solve

$$\max u(x_0, x_1, \ldots, x_{k-1}) = \max \sum_{t=0}^{k-1} \beta^t \ln(x_t) \text{ s.t. } \sum_{t=0}^{k-1} x_t = 1.$$

Note that the natural log function is strictly concave. It is easy to show that then the sum of log functions is also strictly concave (just take the definition of strict concavity and use it in each term of the summation to show that the sum also satisfies strict concavity). Thus, the objective function is strictly concave. Furthermore, the constraint set is compact. Hence, a unique maximum exists and it will be identified by the Lagrangian first order conditions.

$$\mathcal{L}(x_0, x_1, \ldots, x_{k-1}; \lambda) = \sum_{t=0}^{k-1} \beta^t \ln(x_t) + \lambda(1 - \sum_{t=0}^{k-1} x_t)$$

$\overset{(FOC)}{\Rightarrow}$ for all $t \in \{0, \ldots, k-1\}$, $\beta^t \frac{1}{x^t} = \lambda \Rightarrow \lambda \neq 0$.

$\overset{(FOC)}{\Rightarrow}$ $\sum_{t=0}^{k-1} x_t = 1$.

So, for all $t, \bar{t} \in \{0, \ldots, k-1\}$, $\frac{\beta^t}{x^t} = \frac{\beta^{\bar{t}}}{x^{\bar{t}}} \Rightarrow x^t = \frac{\beta^t}{\beta^{\bar{t}}} x^{\bar{t}}$.

Thus,

$$\begin{aligned}\sum_{t=0}^{k-1} x_t &= 1 \Leftrightarrow \sum_{t=0}^{k-1} \frac{\beta^t}{\beta^{\bar{t}}} x^{\bar{t}} = 1 \Leftrightarrow \frac{x^{\bar{t}}}{\beta^{\bar{t}}} \sum_{t=0}^{k-1} \beta^t = 1 \overset{\left[\text{Using } \sum_{t=0}^{k-1} \beta^t = \frac{1-\beta^k}{1-\beta}\right]}{\Longleftrightarrow} \\ \frac{x^{\bar{t}}}{\beta^{\bar{t}}} \frac{1-\beta^k}{1-\beta} &= 1 \Leftrightarrow x^{\bar{t}} = \beta^{\bar{t}} \frac{1-\beta}{1-\beta^k}.\end{aligned}$$

Hence, if the consumer lives k periods, the optimal consumption in period $\bar{t}$ equals $x^{\bar{t}} = \beta^{\bar{t}} \frac{1-\beta}{1-\beta^k}$.

Finally, let $k \to \infty$. Since $\beta^k \underset{k\to\infty}{\longrightarrow} 0$, the optimal consumption in any period t equals $x^t = \beta^t(1-\beta)$.

1.41 (a) The first statement we consider is that a decrease in the own price of a normal good will cause quantity demanded to increase. Formally, as discussed and proven in class,

$$\frac{\partial x_i(p,y)}{\partial y} > 0 \Rightarrow \frac{\partial x_i(p,y)}{\partial p_i} < 0. \tag{1}$$

So, we have to show that

$$\frac{\partial x_i(p,y)}{\partial p_i} < 0 \nRightarrow \frac{\partial x_i(p,y)}{\partial y} > 0. \tag{2}$$

(b) The second statement we consider is that a Giffen good must be an inferior good. Formally, as discussed and proven in class,

$$\frac{\partial x_i(p,y)}{\partial p_i} > 0 \Rightarrow \frac{\partial x_i(p,y)}{\partial y} < 0. \tag{3}$$

So, we have to show that an inferior good is not necessarily a Giffen good.

$$\frac{\partial x_i(p,y)}{\partial y} < 0 \nRightarrow \frac{\partial x_i(p,y)}{\partial p_i} > 0. \tag{4}$$

In both cases a counter example will show that the converse of the statements are not true. A Marshallian demand function that is associated with an inferior good such that (a) we have a negative own price effect and (b) the good is not Giffen will do the job. If you think about the definition of an inferior good, it is clear that first the Marshallian demand for the good has to be increasing in income, but then after some income level, the demand is decreasing if income increases further and the good becomes inferior. Without loss of generality, assume that $p_1 \leq p_2$.

$$\begin{aligned} \text{If } y &\leq p_2, \text{ then } x_1(p,y) = \frac{y}{p_1} \text{ and } x_2(p,y) = 0. \\ \text{If } y &\geq p_2, \text{ then } x_1(p,y) = \frac{p_2}{p_1+y} \text{ and } x_2(p,y) = \frac{y - p_1 x_1(p,y)}{p_2}. \end{aligned}$$

Note that the defined Marshallian demand satisfies homogeneity of degree 0 and budget balancedness (also $x_1(p,y) \geq 0$ and $p_1 x_1(p,y) \leq y$). We have that

$$\frac{\partial x_1(p,y)}{\partial y} = \frac{-p_2}{(p_1+y)^2} < 0 \text{ for } y \geq p_2.$$

Hence, for $y \geq p_2$ good 1 is inferior. Furthermore,

$$\frac{\partial x_1(p,y)}{\partial p_1} = \frac{-p_2}{(p_1+y)^2} < 0 \text{ for } y \geq p_2.$$

This proves that the converse of (1) is not true. Equivalently, it proves (2).

It also proves that the converse of (3) is not true. Equivalently, it proves (4).

1.52 **(a)** Agent A and B's expenditure functions are $e^A(p,u)$ and $e^B(p,u) = e^A(p,2u)$. Is the observable market behavior of the two agents identical? Yes! Agent A and B have identical preferences. The only difference if agent A has utility representation u, then agent B has a utility representation v such that $v(x) = 2u(x)$.

Formally, consider $v^A(p, e^A(p,u)) = u$ and $v^B(p, e^B(p,u)) = v^B(p, e^A(p,2u)) = u$. Thus, $v^B(p, e^A(p,u)) = \frac{u}{2}$ and for all $y = e^A(p,u)$, $v^A(p,y) = 2v^B(p,y)$. By Roy's Identity:

$$\begin{aligned} x_i^A(p,y) &= \frac{\partial v^A(p,y)/\partial p_i}{\partial v^A(p,y)/\partial y} \text{ and} \\ x_i^B(p,y) &= \frac{\partial v^B(p,y)/\partial p_i}{\partial v^B(p,y)/\partial y} \\ &= \frac{\partial(2v^A(p,y))/\partial p_i}{\partial(2v^A(p,y))/\partial y} \\ &= \frac{\partial v^A(p,y)/\partial p_i}{\partial v^A(p,y)/\partial y}. \end{aligned}$$

Hence, $x_i^A(p,y) = x_i^B(p,y)$.

1.53 Let

$$u(x) = A\prod_{i=1}^{n} x_i^{\alpha_i} \text{ s.t. } A, \alpha_1, \ldots, \alpha_n > 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1.$$

(a) Note that if ant $x_i = 0$, then $u(x) = 0$. Hence, at the maximum, we must have $x_i > 0$ for all i. Furthermore, instead of $u(x)$ we could use $v(x) = ln(u(x)) = \ln A + \sum_{i=1}^{n} \alpha_i \ln(x_i)$. This function can easily be shown to be strictly concave. This and the compactness of the constraint set imply that a unique maximizer exists and the Lagrangian first order conditions are sufficient. Since the maximizer for both utility functions will be the same, we can just solve the FOC[s] for $\mathcal{L}(x,\lambda) = A\prod_{i=1}^{n} x_i^{\alpha_i} + \lambda(y - \sum_{i=1}^{n} p_i x_i)$.

$\overset{(FOC)}{\Rightarrow}$ for all i, $A\alpha_i x_i^{\alpha_i - 1}\prod_{k\neq i} x_k^{\alpha_k} = \lambda p_i \Rightarrow \lambda \neq 0$.

$\Rightarrow$ for all i, $A\alpha_i x_i^{-1}\prod_{k=1}^{n} x_k^{\alpha_k} = \lambda p_i$

$\overset{(FOC)}{\Rightarrow}$ $\sum_{i=1}^{n} p_i x_i = 1$.

So, for all i, j,

$$\frac{A\alpha_i x_i^{-1}\prod_{k=1}^{n} x_k^{\alpha_k}}{A\alpha_j x_j^{-1}\prod_{k=1}^{n} x_k^{\alpha_k}} = \frac{\lambda p_i}{\lambda p_j} \Leftrightarrow \frac{\alpha_i x_i^{-1}}{\alpha_j x_j^{-1}} = \frac{p_i}{p_j} \Leftrightarrow x_i = \frac{p_j}{p_i}\frac{\alpha_i}{\alpha_j} x_j.$$

Thus,

$$\begin{aligned} \sum_{i=1}^{n} p_i x_i &= y \Leftrightarrow \sum_{i=1}^{n} p_i \frac{p_j}{p_i}\frac{\alpha_i}{\alpha_j} x_j = y \Leftrightarrow \sum_{i=1}^{n} {}_i \frac{\alpha_i}{\alpha_j} p_j x_j = y \Leftrightarrow \\ \frac{p_j x_j}{\alpha_j}\sum_{i=1}^{n} \alpha_i &= y \overset{\sum_{i=1}^{n}\alpha_i = 1}{\Leftrightarrow} \frac{p_j x_j}{\alpha_j} = y \Leftrightarrow x_j = \frac{\alpha_j y}{p_j}. \end{aligned}$$

Thus, the Marshallian demand function equals

$$x(p, y) = (\frac{\alpha_1 y}{p_1}, \ldots, \frac{\alpha_n y}{p_n}).$$

(b) Using the result from part (a), we obtain,

$$v(p, y) = A\prod_{i=1}^{n}\left(\frac{\alpha_i y}{p_i}\right)^{\alpha_i} = Ay\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}.$$

(c) Recall that $v(p, e(p, u)) = u$. Hence,

$$\begin{aligned} Ae(p,u)\prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} &= u \Leftrightarrow \\ e(p,u) &= \frac{u}{A}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}. \end{aligned}$$

(d) Recall that $x_j^h(p, u) = x_j(p, e(p, u)$. Hence,

$$x_j^h(p,u) = \frac{\alpha_j}{p_j}e(p,u) \Leftrightarrow x_j^h(p,u) = \frac{\alpha_j}{p_j}\frac{u}{A}\prod_{i=1}^{n}\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}.$$

1.60 Let $u^* = v(p, y)$. Show that Slutsky's equation

$$\frac{\partial x_i(p,y)}{\partial p_j} = \frac{\partial x_i^h(p,u^*)}{\partial p_j} - x_j(p,y)\frac{\partial x_i(p,y)}{\partial y}$$

can be expressed in elasticity form as

$$\epsilon_{ij} = \epsilon_{ij}^h - s_j\eta_i. \tag{5}$$

Equation (5) is equivalent to

$$\begin{aligned} \frac{\partial x_i(p,y)}{\partial p_j}\frac{p_j}{x_i(p,y)} &= \frac{\partial x_i^h(p,u^*)}{\partial p_j}\frac{p_j}{x_i^h(p,u^*)} - \frac{p_j x_j(p,y)}{y}\frac{\partial x_i(p,y)}{\partial y}\frac{y}{x_i(p,y)} \Leftrightarrow \\ \frac{\partial x_i(p,y)}{\partial p_j} &= \frac{x_i(p,y)}{p_j}\left[\frac{\partial x_i^h(p,u^*)}{\partial p_j}\frac{p_j}{x_i^h(p,u^*)} - \frac{p_j x_j(p,y)}{y}\frac{\partial x_i(p,y)}{\partial y}\frac{y}{x_i(p,y)}\right] \Leftrightarrow \\ \frac{\partial x_i(p,y)}{\partial p_j} &= \frac{\partial x_i^h(p,u^*)}{\partial p_j}\frac{x_i(p,y)}{x_i^h(p,u^*)} - x_j(p,y)\frac{\partial x_i(p,y)}{\partial y} \overset{x_i(p,y)=x_i^h(p,u^*)}{\Longleftrightarrow} \\ \frac{\partial x_i(p,y)}{\partial p_j} &= \frac{\partial x_i^h(p,u^*)}{\partial p_j} - x_j(p,y)\frac{\partial x_i(p,y)}{\partial y} \text{ (Slutsky Equation).} \end{aligned}$$