4. Homework, Econ 973,
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2.3	\ 5
2.5	\ 11
2.6	\ 5
2.7	\ 4
2.10	\ 4
2.12	\ 3
2.23	\ 7
2.25	\ 2
2.26	\ 5
2.29	\ 4
$\sum$	\ 50

4. Homework, Econ 973,

2.3 Let $v(p, y) = yp_1^\alpha p_2^\beta$ where $\alpha, \beta < 0$. Also note that any indirect utility function should be homogeneous of degree zero. Hence, $\alpha + \beta = -1$. In order to find a direct utility function, solve

$$u(x) = \min_{p \gg 0} v(p, 1) = \min_{p \gg 0} yp_1^\alpha p_2^\beta \text{ s.t. } px = 1.$$

$$\mathcal{L}(p, \lambda) = p_1^\alpha p_2^\beta + \lambda(1 - p_1x_1 + p_2x_2) \overset{(FOC)}{\Rightarrow}$$

$$\begin{aligned} \alpha p_1^{\alpha-1} p_2^\beta &= \lambda x_1 && (1) \\ \beta p_1^\alpha p_2^{\beta-1} &= \lambda x_2 && (2) \\ p_1x_1 + p_2x_2 &= 1. && (3) \end{aligned}$$

(1) and (2) imply that $\lambda, x_1, x_2 \neq 0$ and

$$\begin{aligned} \frac{\alpha p_1^{\alpha-1} p_2^\beta}{x_1} &= \frac{\beta p_1^\alpha p_2^{\beta-1}}{x_2} \Leftrightarrow \\ \frac{\alpha p_2}{x_1} &= \frac{\beta p_1}{x_2} \Leftrightarrow \\ p_1 &= \frac{\alpha x_2}{\beta x_1} p_2. \end{aligned}$$

Now, by (3),

$$\begin{aligned} 1 &= x_1 \left(\frac{\alpha x_2}{\beta x_1} p_2 \right) + p_2x_2 \Leftrightarrow \\ 1 &= \frac{\alpha}{\beta} p_2x_2 + p_2x_2 \Leftrightarrow \\ 1 &= \frac{\alpha + \beta}{\beta} x_2p_2 \Leftrightarrow \\ p_2^* &= \frac{\beta}{(\alpha + \beta)x_2} \left[= \frac{-\beta}{x_2} \right], \\ p_1^* &= \frac{\alpha x_2}{\beta x_1} p_2^* \\ &= \frac{\alpha x_2}{\beta x_1} \frac{\beta}{(\alpha + \beta)x_2} \\ &= \frac{\alpha}{(\alpha + \beta)x_1} \left[= \frac{-\alpha}{x_1} \right]. \end{aligned}$$

Note that $v(p,y) = yp_1^{\alpha}p_2^{\beta}$ is a strictly convex function in (p_1, p_2).

Hence, the FOC$^{\text{s}}$ are sufficient conditions for local and global minima.

$$\begin{aligned} u(x) &= v(p^*, 1) \\ &= \left(\frac{\beta}{(\alpha+\beta)x_2}\right)^{\alpha}\left(\frac{\alpha}{(\alpha+\beta)x_1}\right)^{\beta} \\ &= \frac{\beta^{\alpha}\alpha^{\beta}}{(\alpha+\beta)^{(\alpha+\beta)}x_1^{\alpha}x_2^{\beta}} \left[= -\frac{\beta^{\alpha}\alpha^{\beta}}{x_1^{\alpha}x_2^{\beta}}\right] \text{ (Cobb-Douglas).} \end{aligned}$$

2.5 Let $e(p,u) = up_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$ such that $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

(a) $e(p, v(p,y)) = v(p,y)p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3} = y$. Hence, since $p \gg 0$,

$$v(p,y) = p_1^{-\alpha_1}p_2^{-\alpha_2}p_3^{-\alpha_3}y.$$

Next, we verify Roy's identity:

$$x_i(p,y) = \frac{-\frac{\partial v(p,y)}{\partial p_i}}{\frac{\partial v(p,y)}{\partial y}}.$$

For $i = 1$:

$$\begin{aligned} x_1(p,y) &= \frac{-\frac{\partial v(p,y)}{\partial p_1}}{\frac{\partial v(p,y)}{\partial y}} \\ &= \frac{-\left[-\alpha_1 p_1^{-\alpha_1-1}p_2^{-\alpha_2}p_3^{-\alpha_3}y\right]}{p_1^{-\alpha_1}p_2^{-\alpha_2}p_3^{-\alpha_3}} \\ &= \frac{\alpha_1}{p_1}y. \end{aligned}$$

Similarly, we can show that $x_2(p,y) = \frac{\alpha_2}{p_2}y$ and $x_3(p,y) = \frac{\alpha_3}{p_3}y$ which are the Marshallian demands according to Example 2.3.

(b) By construction

$$\begin{aligned}
u(x) &= \max\{u \in \mathcal{U} \mid x \in A(u)\} \\
&= \max\{u \in \mathcal{U} \mid \text{ for all } p \gg 0,\, px \geq e(p,u)\} \\
&= \max\{u \in \mathcal{U} \mid \text{ for all } p \gg 0,\, px \geq u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}\} \\
&= \max\{u \in \mathcal{U} \mid \text{ for all } p \gg 0,\, \frac{px}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}} \geq u\} \\
&= \max\{u \in \mathcal{U} \mid \min_{p \gg 0} \frac{px}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}} \geq u\} \\
&= \min_{p \gg 0} \frac{px}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}.
\end{aligned}$$

Thus, if $\min_{p \gg 0} \frac{px}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}$ has a solution, then $u(x) = \min_{p \gg 0} \frac{px}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}$.

First, we consider x such that $x_1, x_2, x_3 \neq 0$. $\overset{(FOC\ \frac{\partial}{\partial p_1})}{\Rightarrow}$

$$\begin{aligned}
0 &= \frac{p_1^{\alpha_1} x_1 - \alpha_1 p_1^{\alpha_1 - 1} px}{p_1^{2\alpha_1}} \Leftrightarrow \\
p_1^{\alpha_1} x_1 &= \alpha_1 p_1^{\alpha_1 - 1} px \Leftrightarrow \\
p_1^* &= \frac{\alpha_1 p^* x}{x_1}.
\end{aligned}$$

Similarly, we get

$$p_2^* = \frac{\alpha_2 p^* x}{x_2} \text{ and } p_3^* = \frac{\alpha_3 p^* x}{x_3}.$$

Hence,

$$\begin{aligned}
u(x) &= \frac{p^* x}{(p_1^*)^{\alpha_1} (p_2^*)^{\alpha_2} (p_3^*)^{\alpha_3}} \\
&= \frac{p^* x}{\left(\frac{\alpha_1 p^* x}{x_1}\right)^{\alpha_1} \left(\frac{\alpha_2 p^* x}{x_2}\right)^{\alpha_2} \left(\frac{\alpha_3 p^* x}{x_3}\right)^{\alpha_3}} \\
&= \frac{x_1^{\alpha_1} x_2{}^{\alpha_2} x_3{}^{\alpha_3} p^* x}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3{}^{\alpha_3} (p^* x)^{\alpha_1 + \alpha_2 + \alpha_3}} \\
&= \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3} > 0 \text{ (Cobb-Douglas)}.
\end{aligned}$$

Obviously, $u(x) = 0$ if for some i, $x_i = 0$.

To prove that our utility function $u(x)$ indeed generates the demand function

$x(p, y) = \left(\frac{\alpha_1}{p_1}y, \frac{\alpha_2}{p_2}y, \frac{\alpha_3}{p_3}y\right)$ we solve

$$\max_{x \in \mathbb{R}^n_+} u(x) = \max_{x \in \mathbb{R}^n_+} \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3} \text{ s.t. } px = y.$$

If x is such that some $x_i = 0$, then $u(x) = 0$ which is obviously not the maximal utility level. Hence, when solving for a maximizer, we can assume that $x_i \neq 0$.

$\mathcal{L}(p, \lambda) = \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3} + \lambda(1 - p_1x_1 + p_2x_2 + p_3x_3) \overset{(FOC)}{\Rightarrow}$

$$\begin{aligned}
\alpha_1 \frac{1}{x_1} \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3} &= \lambda p_1 && (4)\\
\alpha_2 \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \frac{1}{x_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3} &= \lambda p_2 && (5)\\
\alpha_3 \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3} \frac{1}{x_3} &= \lambda p_3 && (6)\\
p_1x_1 + p_2x_2 + p_3x_3 &= y. && (7)
\end{aligned}$$

(4), (5), and (6) imply that $\lambda \neq 0$. Then,by (4) and (5),

$$\begin{aligned}
\frac{\alpha_1 \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \frac{1}{x_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3}}{\alpha_2 \left(\frac{x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{x_2}{\alpha_2}\right)^{\alpha_2} \frac{1}{x_2} \left(\frac{x_3}{\alpha_3}\right)^{\alpha_3}} &= \frac{\lambda p_1}{\lambda p_2} \Leftrightarrow\\
\frac{\alpha_1 x_2}{\alpha_2 x_1} &= \frac{p_1}{p_2} \Leftrightarrow\\
x_2 &= \frac{\alpha_2 p_1}{\alpha_1 p_2} x_1.
\end{aligned}$$

Similarly, by (4) and (6), it follows that

$$x_3 = \frac{\alpha_3 p_1}{\alpha_1 p_3} x_1.$$

Hence, by (7),

$$\begin{aligned} y &= p_1x_1 + p_2\frac{\alpha_2 p_1}{\alpha_1 p_2}x_1 + p_3\frac{\alpha_3 p_1}{\alpha_1 p_3}x_1 \Leftrightarrow \\ y &= p_1x_1 + \frac{\alpha_2}{\alpha_1}p_1x_1 + \frac{\alpha_3}{\alpha_1}p_1x_1 \Leftrightarrow \\ y &= \left(1+\frac{\alpha_2}{\alpha_1}+\frac{\alpha_3}{\alpha_1}\right)p_1x_1 \Leftrightarrow \\ x_1^* &= \frac{\alpha_1}{p_1\left(\alpha_1+\alpha_2+\alpha_3\right)} = \frac{\alpha_1}{p_1}y. \end{aligned}$$

Similarly, we obtain,

$$x_2^* = \frac{\alpha_2}{p_2}y \text{ and } x_3^* = \frac{\alpha_3}{p_3}y.$$

2.6 Let $e(p,u) = u\frac{p_1p_2}{p_1+p_2}$.

$$\begin{aligned} u(x) &= \max\{u \in \mathcal{U} \mid x \in A(u)\} \\ &= \max\{u \in \mathcal{U} \mid \text{ for all } p \gg 0,\, px \geq e(p,u)\} \\ &= \max\{u \in \mathcal{U} \mid \text{ for all } p \gg 0,\, px \geq u\frac{p_1p_2}{p_1+p_2}\} \\ &= \max\{u \in \mathcal{U} \mid \text{ for all } p \gg 0,\, \frac{px(p_1+p_2)}{p_1p_2} \geq u\} \\ &= \max\{u \in \mathcal{U} \mid \min_{p\gg 0}\frac{(p_1x_1+p_2x_2)(p_1+p_2)}{p_1p_2} \geq u\} \\ &= \min_{p\gg 0}\left[\frac{p_1x_1+p_2x_2}{p_2}+\frac{p_1x_1+p_2x_2}{p_1}\right]. \end{aligned}$$

Thus, if $\min_{p\gg 0}\left[\frac{p_1x_1+p_2x_2}{p_2}+\frac{p_1x_1+p_2x_2}{p_1}\right]$ has a solution,

then $u(x) = \min_{p\gg 0}\left[\frac{p_1x_1+p_2x_2}{p_2}+\frac{p_1x_1+p_2x_2}{p_1}\right]. \overset{(FOC)}{\Rightarrow}$

$$\begin{aligned} \frac{x_1}{p_2}+\frac{-p_2x_2}{p_1^2} &= 0 \text{ and } \frac{-p_1x_1}{p_2^2}+\frac{x_2}{p_1} = 0 \Rightarrow \\ \frac{x_1}{p_2^2} &= \frac{x_2}{p_1^2} \Leftrightarrow p_1^* = p_2^*\sqrt{\frac{x_2}{x_1}}. \end{aligned}$$

So,

$$\begin{aligned} u(x) &= \frac{p_1^* x_1 + p_2^* x_2}{p_2^*} + \frac{p_1^* x_1 + p_2^* x_2}{p_1^*} \\ &= \frac{p_2^* \sqrt{\frac{x_2}{x_1}} x_1 + p_2^* x_2}{p_2^*} + \frac{p_2^* \sqrt{\frac{x_2}{x_1}} x_1 + p_2^* x_2}{p_2^* \sqrt{\frac{x_2}{x_1}}} \\ &= \sqrt{x_1 x_2} + x_2 + x_1 + \sqrt{x_1 x_2} \\ &= \left(\sqrt{x_1} + \sqrt{x_2}\right)^2 \text{ (CES).} \end{aligned}$$

Alternatively, use the indirect utility function $v(p, y)$ associated with $e(p, u)$ to find $u(x)$.

2.7 Let $u(x) = A x_1^{\alpha} x_2^{1-\alpha}$; $\alpha \in (0, 1)$. By Hotelling Wold,

$$p_i(x) = \frac{\frac{\partial u(x)}{\partial x_i}}{\sum_{j=1}^{2} x_j \frac{\partial u(x)}{\partial x_j}}.$$

We have

$$\begin{aligned} \frac{\partial u(x)}{\partial x_1} &= \alpha A x_1^{\alpha-1} x_2^{1-\alpha}, \\ \frac{\partial u(x)}{\partial x_2} &= (1-\alpha) A x_1^{\alpha} x_2^{-\alpha}, \\ \frac{\partial u(x)}{\partial x_1} x_1 + \frac{\partial u(x)}{\partial x_2} x_2 &= \alpha A x_1^{\alpha} x_2^{1-\alpha} + (1-\alpha) A x_1^{\alpha} x_2^{1-\alpha} \\ &= A x_1^{\alpha} x_2^{1-\alpha}. \end{aligned}$$

So,

$$\begin{aligned} p_1(x) &= \frac{\alpha A x_1^{\alpha-1} x_2^{1-\alpha}}{A x_1^{\alpha} x_2^{1-\alpha}} = \frac{\alpha}{x_1} \text{ and} \\ p_2(x) &= \frac{(1-\alpha) A x_1^{\alpha} x_2^{-\alpha}}{A x_1^{\alpha} x_2^{1-\alpha}} = \frac{(1-\alpha)}{x_2}. \end{aligned}$$

2.10 WARP: $\tilde{p}\tilde{x} \geq \tilde{p}\bar{x} \Rightarrow \bar{p}\tilde{x} > \bar{p}\bar{x}$. Note that

$$p^0x^0 = 42,\ p^1x^1 = 36, \text{ and } p^2x^2 = 50.$$

(a) Compare x^0, x^1:

Note that $p^0x^1 = 48$ and $p^1x^0 = 33$. Thus,

$$\begin{aligned} p^0x^0 &= 42 \not\geq 48 = p^0x^1 \text{ (WARP satisfied)},\\ p^1x^1 &= 36 \geq 33 = p^1x^0 \text{ and } p^0x^1 = 48 > 42 = p^0x^0 \text{ (WARP satisfied)}. \end{aligned}$$

The second statement implies that x^1 is revealed preferred to x^0, i.e., $x^1 \succ^R x^0$.

Compare x^1, x^2:

Note that $p^1x^2 = 39$ and $p^2x^1 = 48$. Thus,

$$\begin{aligned} p^1x^1 &= 36 \not\geq 39 = p^1x^2 \text{ (WARP satisfied)},\\ p^2x^2 &= 50 \geq 48 = p^2x^1 \text{ and } p^1x^2 = 39 > 36 = p^1x^1 \text{ (WARP satisfied)}. \end{aligned}$$

The second statement implies that x^2 is revealed preferred to x^1, i.e., $x^2 \succ^R x^1$.

Compare x^2, x^0:

Note that $p^2x^0 = 52$ and $p^0x^2 = 40$. Thus,

$$\begin{aligned} p^2x^2 &= 50 \not\geq 52 = p^2x^0 \text{ (WARP satisfied)},\\ p^0x^0 &= 42 \geq 40 = p^0x^2 \text{ and } p^2x^0 = 52 > 50 = p^2x^2 \text{ (WARP satisfied)}. \end{aligned}$$

The second statement implies that x^0 is revealed preferred to x^2, i.e., $x^0 \succ^R x^2$.

(b) We now have that $x^0 \succ^R x^2$ and $x^2 \succ^R x^1$. Hence, transitivity would imply that $x^0 \succ^R x^1$. However, when comparing bundles x^1 and x^0 we found $x^1 \succ^R x^0$. Note that $x^0 \succ^R x^1$ and $x^1 \succ^R x^0$ cannot be true at the same time. Therefore, transitivity is not satisfied.

2.12 We use the following abbreviations for alternatives:

T = theft
NT = no theft
I = insurance
NI = no insurance

The set of possible outcomes equals $A = \{(T, I), (T, NI), (NT, I), (NT, NI)\}$. With insurance, an agent faces the gamble

$$g_I = (p \circ (T, I), 0 \circ (T, NI), (1 - p) \circ (NT, I), 0 \circ (NT, NI))$$

and without insurance

$$g_{NI} = (0 \circ (T, I), p \circ (T, NI), 0 \circ (NT, I), (1 - p) \circ (NT, NI)).$$

Assuming that the agent has total wealth $w \geq D$ (possibly $w = D$) and that the insurance pays D if the asset is stolen, we also have

$$\begin{aligned} g_I &\sim (p \circ (w - I), 0 \circ (w - D), (1 - p) \circ (w - I), 0 \circ w) \\ &\sim (1 \circ (w - I)) \end{aligned}$$

and

$$\begin{aligned} g_{NI} &\sim (0 \circ (w - I), p \circ (w - D), 0 \circ (w - I), (1 - p) \circ w) \\ &\sim (p \circ (w - D), (1 - p) \circ w). \end{aligned}$$

Hence,

$$\begin{aligned} \text{if } u(w - I) &> pu(w - D) + (1 - p)u(w), \text{ then choose the insurance and} \\ \text{if } u(w - I) &< pu(w - D) + (1 - p)u(w), \text{ then choose no insurance.} \end{aligned}$$

2.23 Let $U(w) = a + bw + cw^2$. First we need to assume that $U'(w) = b + 2cw > 0$.

(a) risk aversion $\Rightarrow U''(w) = 2c < 0 \Rightarrow c < 0$. Thus, $b > -2cw > 0$.

(b) Since $U'(w) = b + 2cw > 0 \Rightarrow w < -\frac{b}{2c}$.

(c) Let $g = (\frac{1}{2} \circ (w + h), \frac{1}{2} \circ (w - h))$. Show that $CE < E(g)$ and $P > 0$.

By definition of the certainty equivalent, $u(CE) = u(g)$.

$$\begin{aligned}
u(g) &= \frac{1}{2}[a + b(w + h) + c(w + h)^2] + \frac{1}{2}[a + b(w - h) + c(w - h)^2] \\
&= a + \frac{1}{2}b[w + h + w - h] + \frac{1}{2}c[(w + h)^2 + (w - h)^2] \\
&= a + bw + \frac{1}{2}c[w^2 + 2wh + h^2 + w^2 - 2hw + h^2] \\
&= a + bw + c(w^2 + h^2) \\
u(CE) &= u(g) \\
&= a + bw + c(w^2 + h^2) \\
&< a + bw + cw^2 \\
&= u(w).
\end{aligned}$$

Thus,

$$u(CE) < u(w) \overset{U'>0}{\Leftrightarrow} CE < w.$$

Then,

$$P = E(g) - CE = w - CE > 0.$$

(d) Decreasing absolute risk aversion means that

$$R_a'(w) = \left(-\frac{U''(w)}{U'(w)}\right)' < 0.$$

However, since $b + 2cw > 0$ and $4c^2 > 0$ we have that

$$R_a'(w) = \left(-\frac{2c}{b + 2cw}\right)' = \frac{4c^2}{(b + 2cw)^2} > 0.$$

2.25 Let $\beta > 0$ and $u(w) = \alpha + \beta \ln w$.

$$R_a(w) = -\frac{u''(w)}{u'(w)} = -\frac{-\frac{\beta}{w^2}}{\frac{\beta}{w}} = \frac{1}{w}.$$

Hence,

$$R_a'(w) = -\frac{1}{w^2} < 0$$

which means that $u\,(w)$ displays decreasing absolute risk aversion.

2.26 Let $u(x) = \ln x_1 + 2\ln x_2$. Let $p = (1,1)$. Note that the given utility function is defined over bundles of commodities and not over wealth\income levels. In order to determine the risk attitude of the agent when offered different gambles over amounts of income, we first have to find the indirect utility function associated with p and y.

$$\begin{aligned} \max_{x\in\mathbb{R}^2_+} u(x) \text{ s.t. } p_1x_1 + p_2x_2 &= y \\ \max_{x\in\mathbb{R}^2_+} x &\in \mathbb{R}^2_+ \ln x_1 + 2\ln x_2 \text{ s.t. } x_1 + x_2 = y. \end{aligned}$$

Note that at a maximum $x_1 \neq 0$ and $x_2 \neq 0$.

$\mathcal{L}(p,\lambda) = \ln x_1 + 2\ln x_2 + \lambda(y - x_1 - x_2) \overset{(FOC)}{\Rightarrow}$

$$\frac{1}{x_1} = \lambda,\ \frac{2}{x_2} = \lambda, \text{ and } x_1 + x_2 = y.$$

Hence, $\lambda \neq 0$ and $\frac{1}{x_1} = \frac{2}{x_2} \Rightarrow x_2 = 2x_1$.

Thus, $x_1 + 2x_1 = y \Rightarrow x_1 = \frac{y}{3}$ and $x_2 = \frac{2y}{3}$.

So, the agents indirect utility given the prices $p = (1,1)$ equals

$$\begin{aligned} v(y) &= \ln\frac{y}{3} + 2\ln\frac{2y}{3} \\ &= \ln y - \ln 3 + 2[\ln 2 + \ln y - \ln 3] \\ &= 3\ln y + 2\ln 2 - 3\ln 3. \end{aligned}$$

Note that $v'(y) = \frac{3}{y}$ and $v''(y) = -\frac{3}{y^2} < 0$. Hence, the agent will be risk avers.

2.29 Prove that for any VNM utility function $u'''(w) > 0$ is necessary, but not sufficient for DARA.

Necessity To show: DARA $\Rightarrow u'''(w) > 0$.

DARA implies that the agent is risk avers, i.e., $u''(w) < 0$ and $R_a'(w) < 0$.

$$R_a'(w) < 0 \Leftrightarrow \tfrac{\partial}{\partial w}\left(-\tfrac{u''(w)}{u'(w)}\right) = -\tfrac{u'(w)u'''(w)-(u''(w))^2}{(u'(w))^2} < 0 \Leftrightarrow \underbrace{(u''(w))^2}_{\geq 0} - u'(w)u'''(w) < 0.$$

Hence, $-\underbrace{u'(w)}_{>0} u'''(w) < 0$.Thus, $u'''(w) > 0$.

Sufficency Counterexample: $u(w) = w^3$. Then, $u'(w) = 3w^2$, $u''(w) = 6w > 0$, $u'''(w) = 6 > 0$. So, $u'''(w) > 0$, but

$$\begin{aligned} R_a'(w) &= \frac{(u''(w))^2 - u'(w)u'''(w)}{(u'(w))^2} \\ &= \frac{36w^2 - 18w^2}{9w^4} \\ &= \frac{2}{w^2} > 0 \text{ (IARA)}. \end{aligned}$$