© Paul A. Johnson

Economics 304
Advanced Topics in Macroeconomics

Notes on Elementary Optimal Growth Models

A Two-Period Model

We begin with an economy that lasts two periods ("now" and "future") inhabited by a large number of identical individuals. These individuals have preferences over consumption in the two periods described by the lifetime utility function

$$V(c_1, c_2) = U(c_1) + \frac{1}{1+\rho}U(c_2)$$

where c_1 and c_2 are consumption in periods 1 and 2 respectively, and ρ is the subjective rate of time preference. The "instantaneous" utility function, $U(\,\cdot\,)$, is increasing and strictly concave.[1]

The individual has access to a technology described by the production function $f(k)$ which is assumed to be increasing and concave with $f(0) = 0$. The individual begins period 1 with a capital stock of k_1. Letting i_1 denote the amount of saving (investment) done in period 1 we have the constraints

$$i_1 = f(k_1) - c_1$$

$$k_2 = (1-\delta)k_1 + i_1$$

$$c_2 = f(k_2) + (1-\delta)k_2$$

where δ is the depreciation rate. The first equation says that saving is the excess of output over consumption. The second says that the capital stock in period 2 is what is left after depreciation of that in period 1 plus saving. This is sometimes called a "law of motion". The third says that the individual consumes all available resources in period 2. The first two constraints may be combined and solved for c_1 to give

$$c_1 = f(k_1) + (1-\delta)k_1 - k_2.$$

The individual's objective is to maximize their lifetime utility subject to these constraints. One way to solve this problem is to begin by subsituting the constaints into the utility function to give

---

[1] "Increasing" means that $U' > 0$ and "strictly concave" means that $U'' < 0$. We usually also impose the condition $U'(0) = \infty$ to prevent zero consumption in any period.

$$V(k_2) = U(f(k_1) + (1-\delta)k_1 - k_2) + \frac{1}{1+\rho}U(f(k_2) + (1-\delta)k_2).$$

Now take the derivative with respect to k_2 and set it to zero:

$$\frac{dV}{dk_2} = -U'(c_1) + \frac{1}{1+\rho}U'(c_2)[f'(k_2) + (1-\delta)] = 0.$$

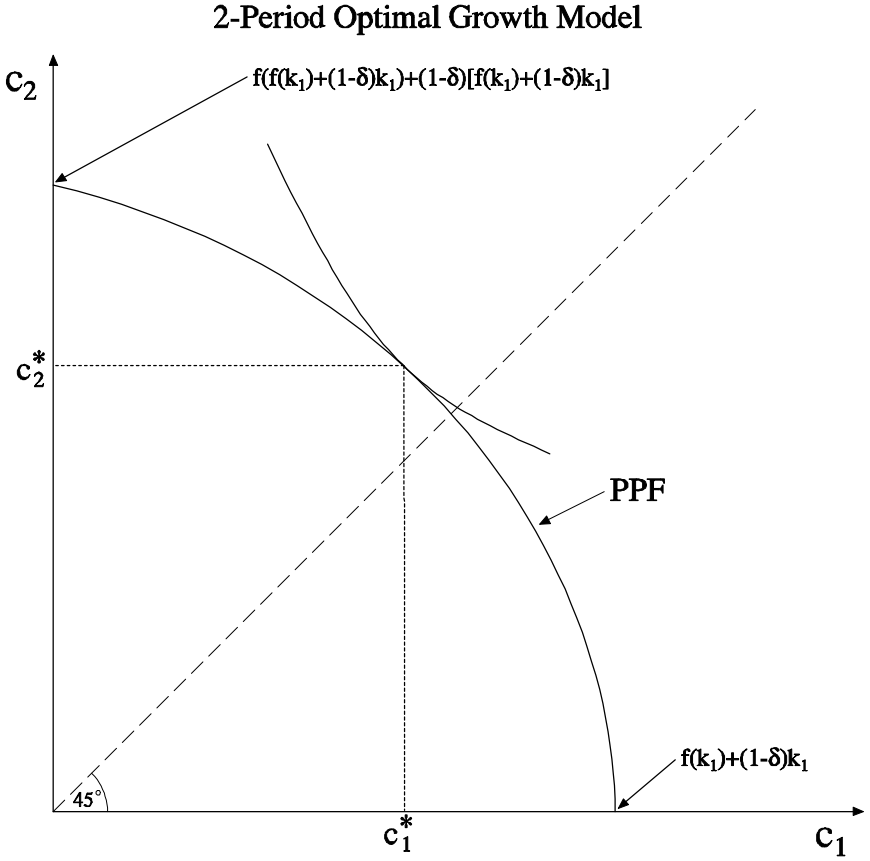
This condition may be rewritten as

$$U'(c_1) = \frac{1}{1+\rho}U'(c_2)[f'(k_2) + (1-\delta)].$$

To see why this is a neccessary condition for optimality consider an individual who foregoes one unit of consumption in period 1. The utility loss is $U'(c_1)$. The unit of consumption becomes an extra unit of capital in period 2 and increases period 2 resouces by $f'(k_2) + (1-\delta)$. This increases period 2 utility by $U'(c_2)[f'(k_2) + (1-\delta)]$ and the present (that is, period 1) value of this increase is $\frac{1}{1+\rho}U'(c_2)[f'(k_2) + (1-\delta)]$. If this gain is greater than the original utility loss this scheme increases lifetime utility. If the gain is less than the loss lifetime utility can be increased by increasing period 1 consumption at the expense of period 2 consumption. Only when the gain equals the loss is lifetime utility at a maximum.

Note that the individual will never choose a level of capital in the second period implying $f'(k_2) < \delta + \rho$. To do so would imply that lifetime utility could be increased by consuming more in the first period and saving less. To see this observe that $f'(k_2) < \delta + \rho$ implies $\frac{1}{1+\rho}[f'(k_2) + (1-\delta)] < 1$. The implication is that, in this situation, increasing first period consumption by one unit increases total utility. We can conclude that the individual would never choose a level of investment implying $c_1 > c_2$.

We can illustrate this on a diagram that is similar to one familar from intermediate microeconomics.



The curved line labeled PPF is the individual's production possibility frontier. It describes how period 2 and period 1 consumption may be transformed from one to the other using the production technology. Using the constraints of the problem the equation of this line is found to be

$$c_2 = f(f(k_1) + (1-\delta)k_1 - c_1) + (1-\delta)[f(k_1) + (1-\delta)k_1 - c_1].$$

From this expression it can be shown that $\left.\frac{dc_2}{dc_1}\right|_{\text{PPF}} = -f'(k_2) - (1-\delta) < 0$ and $\left.\frac{d^2c_2}{dc_1^2}\right|_{\text{PPF}} = f''(k_2) < 0$. Hence the shape of the PPF. The individual's indifference curves have the familiar shape and it can be shown that the slope of an indifference curve is given by $\left.\frac{dc_2}{dc_1}\right|_{dV=0} = -(1+\rho)\frac{U'(c_1)}{U'(c_2)}$. The individual achieves the maximal feasible level of utility by choosing that point on the PPF that is also on the highest possible indifference curve. This is the point (c_1^*, c_2^*) as shown. At that point, the PPF and the indifference curve are tangent to each other so that $\left.\frac{dc_2}{dc_1}\right|_{\text{PPF}} = \left.\frac{dc_2}{dc_1}\right|_{dV=0}$. Manipulation of the resultant condition gives the condition found above using calculus.

A Note on Technology

The two-period model above assumes that the individual has access to a technology described by the production function $y = f(k)$. Generally we will be concerned with economies that use both capital and labor as inputs to production. Letting

Y be total output (GDP), K be the capital stock, and L be the amount of labor input, we will summarize the technology of the economy with the aggregate production function $Y = F(K, L)$. Typically we will assume that F exhibits constant returns to scale so that $\lambda F(K, L) = F(\lambda K, \lambda L)$ for any $\lambda > 0$. If we take $\lambda = \frac{1}{L}$ we have $\frac{F(K,L)}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right)$ so, defining $y = \frac{Y}{L}$, $k = \frac{K}{L}$, and $f(k) = F(k, 1)$ we have $y = f(k)$ which implies that output per unit of labor depends only on capital per unit of labor.

The per capita production function, $y = f(k)$, has the same marginal product of capital as the $Y = F(K, L)$ production function. To see this differentiate $\lambda F(K, L) = F(\lambda K, \lambda L)$ with respect to K to get $\lambda \frac{\partial F(K,L)}{\partial K} = \lambda \frac{\partial F(\lambda K, \lambda L)}{\partial(\lambda K)}$ which implies $\frac{\partial F(K,L)}{\partial K} = \frac{\partial F(\lambda K, \lambda L)}{\partial(\lambda K)}$ so that setting $\lambda = \frac{1}{L}$ we have $\frac{\partial F(K,L)}{\partial K} = \frac{\partial F\left(\frac{K}{L}, 1\right)}{\partial\left(\frac{K}{L}\right)} = f'(k)$.

An important property of constant returns to scale production functions is that, under competition, the payments to factors of production equal total output so (economic) profits are zero. To see this in the general case begin with $\lambda F(K, L) = F(\lambda K, \lambda L)$ which is true for any $\lambda > 0$. Differentiate this expression with respect to λ to get $F(K, L) = \frac{\partial F(\lambda K, \lambda L)}{\partial(\lambda K)} K + \frac{\partial F(\lambda K, \lambda L)}{\partial(\lambda L)} L$ so that if we set $\lambda = 1$ we have $F(K, L) = \frac{\partial F(K,L)}{\partial K} K + \frac{\partial F(K,L)}{\partial L} L$. Under competition, the payment per unit of capital is equal to the marginal product of capital, $\frac{\partial F(K,L)}{\partial K}$, so $\frac{\partial F(K,L)}{\partial K} K$ is the total payment to capital. Similarly, $\frac{\partial F(K,L)}{\partial L} L$ is the total payment to labor so $\frac{\partial F(K,L)}{\partial K} K + \frac{\partial F(K,L)}{\partial L} L$ is the total payments to all factors of production which is shown to equal total output, $F(K, L)$.

The results in the last two paragraphs can be used to calculate the real wage when we use the per capita production function. Let w be the real wage so that $wL = Y - \frac{\partial F(K,L)}{\partial K} K$. Divide both sides by L to get $w = \frac{Y}{L} - \frac{\partial F(K,L)}{\partial K} \frac{K}{L}$ $= f(k) - f'(k)k$ where we have used $\frac{Y}{L} = f(k)$ and $\frac{\partial F(K,L)}{\partial K} = f'(k)$.

Generalizing to Many Periods

Consider now the case where individuals are infinely lived. We index time periods by t so that the law of motion for capital becomes

$$k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t \tag{1}$$

with $k_0 > 0$ given.[2] Individuals continue to maximize lifetime utility which is now given by

$$V = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t U(c_t).$$

We can solve the problem by writing $c_t = (1 - \delta)k_t + f(k_t) - k_{t+1}$ and subsituting into the objective function to get

[2]The Solow growth model has $c_t = (1 - s)f(k_t)$ where s is the (exogenous) saving rate.

$$V = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t U((1-\delta)k_t + f(k_t) - k_{t+1}).$$

Setting the derivative of this expression with respect to k_{t+1} equal to zero and doing a little algebra yields the optimality condition

$$U'(c_t) = \frac{1}{1+\rho} U'(c_{t+1})[f'(k_{t+1}) + (1-\delta)]. \tag{2}$$

The intuition is the same as in the two period case.[3]

The Steady State

The steady state is reached when the capital stock (and hence output) and consumption stop changing. That is, $k_t = \widehat{k}$, and $c_t = \widehat{c}$. Substituting these conditions into the first order conditions gives

$$U'(\widehat{c}) = \frac{1}{1+\rho} U'(\widehat{c})[f'(\widehat{k}) + (1-\delta)]$$

$$\Rightarrow 1 = \frac{1}{1+\rho}[f'(\widehat{k}) + (1-\delta)]$$

$$\Rightarrow f'(\widehat{k}) - \delta = \rho \tag{3}$$

and

$$\widehat{k} = (1-\delta)\widehat{k} + f(\widehat{k}) - \widehat{c}$$

$$\Rightarrow \widehat{c} = f(\widehat{k}) - \delta\widehat{k}. \tag{4}$$

Equation (3) equates the net marginal product of capital, $f'(k) - \delta$, with the rate of time preference. Once this condition is achieved there is no further incentive to accumulate capital as the reward for postponing consumption is just equal to the cost. Once $\widehat{k}$ is determined by equation (3), equation (4) can be used to determine $\widehat{c}$. The steady state level of output is given by $\widehat{y} = f(\widehat{k})$.

Note that $\widehat{k}$ is <u>not</u> the capital stock that maximizes the level of consumption. To find that capital stock differentiate $c = f(k) - \delta k$ with respect to k and set the result equal to zero to conclude $f'(k_g) - \delta = 0$. Where we use the notation k_g to denote the

---

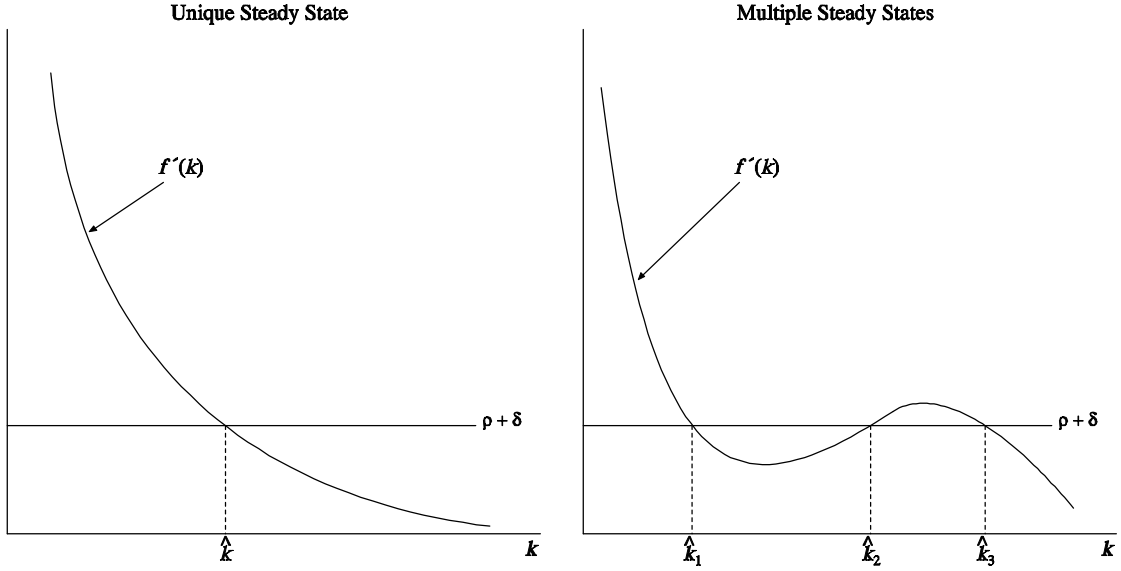
[3]The complete set of first order conditions are the law of motion for capital, the previous equation and a condition known as the "transversality" condition. In this case that condition is $\lim_{t\to\infty} \left(\frac{1}{1+\rho} \right)^t U'(c_t)k_t = 0$ which has the effect of preventing the economy from accumulating "too much" capital. If it were to accumulate "too much" capital c_t eventually begins to fall and the quantity $U'(c_t)k_t$ grows at a rate exceeding ρ which violates this condition. Doing so implies that lifetime utility can be increased by an immediate reduction in the capital stock and an increase in consumption. To avoid this situation, the economy must choose the (unique) initial level of consumption such that, by following the other first-order conditions, it converges to the steady state.

"golden rule" capital stock. Comparison with equation (3) (remembering that $f''(k) < 0$) reveals that $k_g > \widehat{k}$. We conclude that, because individuals have a positive rate of time preference, the golden rule capital stock and level of consumption are not achieved in this model.

Equations (1) and (2) govern the transition of the economy to its steady state given an initial capital stock, k_0, and consumption level, c_0.[4] We can show that if $k_0 < \widehat{k}$, then $k_{t+1} > k_t$ during this transition. The behavior of consumption depends on the intertemporal elasticity of substitution and how it varies, if at all, during the transition. See the example below for a discussion in the case of a constant elasticity.

Note that we need the production function to satisfy the condition $\lim_{k\to\infty} f'(k) \leq \delta + \rho$ to ensure convergence to a steady state. We will actually impose the stronger condition $\lim_{k\to\infty} f'(k) = 0$ which is known as an Inada condition.[5] If this condition is not satisfied, the steady state will not exist. Convergence, and the ceasation of growth, is then caused by the accumulation of capital which drives the marginal product of capital to a point where there is no further incentive to accumulate.

It is important to understand how the asuumptions made about the shape of the production function give the existence and uniqueness of the steady state. We have assumed that $f(k)$ satisfies $f'(k) > 0$, $f''(k) < 0$, $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k) = 0$. The figure below plots $f'(k)$ satisfying these assumptions. The function is always positive and declining. For small values of k $f'(k)$ is very large and for large values of k it is very small. This ensures that $f'(k)$ is larger than $\rho + \delta$ for small values of k, smaller than $\rho + \delta$ for larger values of k, and equal to $\rho + \delta$ for only one value of k, giving a unique steady state at $\widehat{k}$. Moreover, these features ensure that, if k is not equal to $\widehat{k}$, the economy always moves toward the steady state.



[4]The choice of initial consumption requires consideration of the transversality condition so I do not deal with it here. This issue is taken up later.

[5]We will also impose another Inada condition, $\lim_{k\to 0} f'(k) = \infty$, to ensure that some growth occurs.

The marginal product curve of a production function violating some of the assumptions we have made is shown in the other diagram. In particular this function violates the assumption $f''(k) < 0$ in such a way that $f'(k) = \rho + \delta$ at three values of k. Thus there are three steady states. Two of them, $\widehat{k}_1$ and $\widehat{k}_3$, have the property that if the economy is close to either it will move toward the appropriate steady state. For example, if $\widehat{k}_1 < k_t < \widehat{k}_2$ then $\widehat{k}_1 < k_{t+1} < k_t$ because $f'(k) < \rho + \delta$ for $\widehat{k}_1 < k_t < \widehat{k}_2$ implying negative net investment.[6] On the other hand, the steady state at $\widehat{k}_2$ does not have this property so if k is close but not equal to $\widehat{k}_2$ the economy moves away from this point.[7] If a group of economies had this production function we would eventually se them split into two grooups as some moved toward the $\widehat{k}_1$ steady state while the others moved toward the $\widehat{k}_3$ steady state.

An Example

Suppose that the instantaneous utility function is $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ so that $U'(c) = c^{-\sigma}$ with $\sigma > 0$ and that the production function is $f(k) = Ak^{\alpha}$ so that $f'(k) = \alpha A k^{\alpha-1}$ with $0 < \alpha < 1$. Under these assumptions the first order conditions for the problem are

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \frac{1}{1+\rho}[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)]$$

and

$$k_{t+1} = (1-\delta)k_t + Ak_t^{\alpha} - c_t.$$

Given an inital capital stock, k_0, and an initial consumption level, c_0, these equations determine the behavior of the system as it transitions to the steady state. The steady state values of capital and consumption are given by

$$\widehat{k} = \left(\frac{\alpha A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$$

and

$$\widehat{c} = \frac{\rho + \delta(1-\alpha)}{\alpha}\left(\frac{\alpha A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}.$$

The parameter σ is the inverse of the "intertemporal elasticity of subsitution" and it measures the unwillingness of individual's to tolerate changes in their level of consumption over time. We can regard the expression $\frac{\alpha A k_{t+1}^{\alpha-1}+(1-\delta)}{1+\rho}$ as measuring the incentive to postpone consumption in period t and accumulate capital instead. For a given value of this quantity, the growth in consumption between periods t and $t+1$, $\frac{c_{t+1}}{c_t}$, declines as σ

---

[6] We say that the steady states at $\widehat{k}_1$ and $\widehat{k}_3$ are (locally) stable.

[7] We say that the steady state at $\widehat{k}_2$ is unstable.

increases. A higher value of σ reduces the response of the individual to the incentive so that c_t is larger than it otherwise would be and the growth rate is smaller. As the steady-state level of consumption does not depend on σ we can conclude that the speed of convergence to the steady state is declining in σ.

Equilibrium and Optimality

The analysis above treats the (identical) individuals in the economy as if they own and operate capital as well as supply labor. This is equivalent to assuming that there is a social planner who makes the production and consumption decisions on behalf of the individuals with the objective of maximixing their lifetime utility. This should not be interpreted as a model of central planning or anything similar. Rather, it turns out that, under certain conditions, the decisions made by the ficticious social planner are exactly the same as those that would occur in a competitive equilibrium and it is (much) easier to solve the social planners problem than compute a competitive equilibrium.[8]

To see the coincidence of the solution to the social planner's choices and the competitive equilibrium suppose that the household sector of the economy consists of households who rent k_t units of capital to firms at the price r_{t+1} and inellastically supply one unit of labor each period to firms at the price w_t. Households and firms take these prices as given. The households divide their income, $r_{t+1}k_t + w_t$, between consumption, c_t, and the accumulation of capital so that $k_{t+1} - k_t = r_{t+1}k_t + w_t - c_t$, in order to maximize lifetime utility $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t U(c_t)$. This is exactly the same problem as we studied in the section on intertemporal consumption choice so we know that the optimality condtion is $U'(c_t) = \frac{1+r_{t+1}}{1+\rho}U'(c_{t+1})$.

The firms in this economy use the capital and labor rented from households to produce output which they sell to households. Our assumption of constant returns to scale ensures that profits are zero so we don't have to worry about stocks and dividends. The production function is $f(k)$ and since there is only one good, the relative price of capital is zero, and we know that the firm will rent capital to the point where $f'(k_t) = r_{t+1} + \delta$.

Combining the firm and household optimality conditions gives the social planner's optimality condition, equation (2) above. This implies that social planner's choices and the outcomes in a competitive equilibrium are the same. We need to be careful in applying this principal as in many cases it does not hold. We shall encounter some such cases in our study of endogenous growth models later.

Population Growth

So far we have considered the case of an economy with a constant population. The generalization to the case of an economy with a population that grows at an exogenous, constant rate is quite straightforward. Let the population at time t be denoted

[8]The coincidence between competitive equilibria and optimal allocations is the subject of the two fundamental theorems of welfare economics.

by L_t and assume that it grows at the rate n so that $L_t = (1+n)L_{t-1}$ or $L_t = (1+n)^t L_0$ where L_0 is the given initial population.[9]

Define all of the variables y, k, and c to be in per capita terms so that, for example, y is now GDP per person. The law of motion for the per capita capital stock ("capital per worker" or "the capital-labor ratio") is $k_{t+1} = (1-\delta-n)k_t + f(k_t) - c_t$. The only difference is the inclusion of the rate of population growth which enters the expression in the same way as the depreciation rate.[10]

The appropriate objective function for the social planner is $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t L_t U(c_t)$. Note that this is the same as for the case with a constant population except that we weight the utility of the representative individual at time t by the number of individuals in the population at that time.[11] Using the expression $L_t = (1+n)^t L_0$ we can write this objective as $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t (1+n)^t L_0 U(c_t) = L_0 \sum_{t=0}^{\infty}\left(\frac{1+n}{1+\rho}\right)^t U(c_t)$ where we will assume $n < \rho$ to ensure that the sum is finite. We will also ignore the L_0 term as it does not effect the choice made by the social planner.[12]

Maximization of this objective subject to the constraints imposed by the law of motion for capital yields the optimality condition

$$U'(c_t) = \frac{1+n}{1+\rho} U'(c_{t+1})[f'(k_{t+1}) + (1-\delta-n)]$$

so that one of the expressions defining the steady state is

---

[9]Note that we are treating the population and the workforce as if they were the same. This has no consequence for the generality of our conclusions provided the latter is a constant fraction of the former.

[10]To see why population growth acts like depreciation suppose that there are 100,000 units of capital and 1,000 workers in period t so that there are 100 units of capital per worker. If the workforce grows by 2% it is 1020 = 1000(1.02) in period $t+1$ so, absent any investment, the amount of capital per worker is now 98. This is the same as the amount of capital per worker that would exist in period $t+1$ if the number of workers had remained constant and the capital stock depreciated by 2% to 98,000 units.

[11]The other possible objective function is $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t U(c_t)$. To see why I have used that in the text consider the problem of allocating X units of a good between the residents of two islands with populations L_1 and L_2 respectively. Assume that the residents have the same utility function $U(x)$ where x is the amount of the good that they receive. One way to decide how much to allocate to each individual is to choose to give X_1 units to island 1 and X_2 units to island 2 to maximize $L_1 U\left(\frac{X_1}{L_1}\right) + L_2 U\left(\frac{X_2}{L_2}\right)$ subject to the restriction $X_1 + X_2 = X$. The solution to this problem gives each individual the same amount of the good. That is, the solution is $X_1 = \frac{L_1}{L_1+L_2}X$ and $X_2 = \frac{L_2}{L_1+L_2}X$ so that $\frac{X_1}{L_1} = \frac{X_2}{L_2}$. Note the analogy between the objective function here and that in the text. The objective function analogous to that introduced at the beginning of this footnote is $U\left(\frac{X_1}{L_1}\right) + U\left(\frac{X_2}{L_2}\right)$. Maximization of this objective subject to $X_1 + X_2 = X$ implies giving more of the good per person to those on the least populous island. That is, if $L_1 > L_2$ then $\frac{X_1}{L_1} < \frac{X_2}{L_2}$. It is my view that the egalitarian allocation produced by the earlier objective is preferable. Hence the objective function used in the text.

[12]The x that maximizes $f(x)$ also maximizes $af(x)$ for any $a > 0$.

$$U'(\widehat{c}) = \frac{1+n}{1+\rho}U'(\widehat{c})[f'(\widehat{k}) + (1-\delta-n)]$$

which implies the condition $f'(\widehat{k}) - \delta = \rho$ as before.[13] The other condition becomes $\widehat{c} = f(\widehat{k}) - (n+\delta)\widehat{k}$. Setting the derivative of $c = f(k) - (n+\delta)k$ with respect to k equal to zero shows that the capital stock maximizing per capita consumption is the solution to $f'(k_g) = n + \delta < \rho + \delta = f'(\widehat{k})$. So, as before, the steady-state capital stock is less than that maximizing consumption per capita.

Transition Dynamics

We have studied the steady state but we have yet to say anyhing about how the economy gets there. We call this period the "transition" and, as we shall see below, depending on the parameter values it can be quite long. Ideally our study of this phase would produce an equation describing the behavior of y over time. This is, however, often an unattainable task so we use a device known as a phase diagram.

The law of motion of capital can be written as

$$k_{t+1} - k_t = \Delta k_{t+1} = f(k_t) - (n+\delta)k_t - c_t$$

so that $\Delta k_{t+1} = 0$ implies $c_t = f(k_t) - (n+\delta)k_t$. Any (c_t, k_t) pair satisfying this equation will imply that there is no change in the capital stock. We want to draw this locus on a diagram with k_t on the horizontal axis and c_t on the vertical axis. Note that the line will pass through the origin, have a positive slope for small values of k_t, reach a maximum at $k_t = k_g$, and have a negative slope for $k_t > k_g$. This line is shown on the phase diagram below. For points above this line $c_t > f(k_t) - (n+\delta)k_t$ so that $\Delta k_{t+1} < 0$ as indicated by the left-pointing horizontal arrows above the line. For points below this line $c_t < f(k_t) - (n+\delta)k_t$ so that $\Delta k_{t+1} > 0$ as indicated by the right-pointing horizontal arrows below the line.

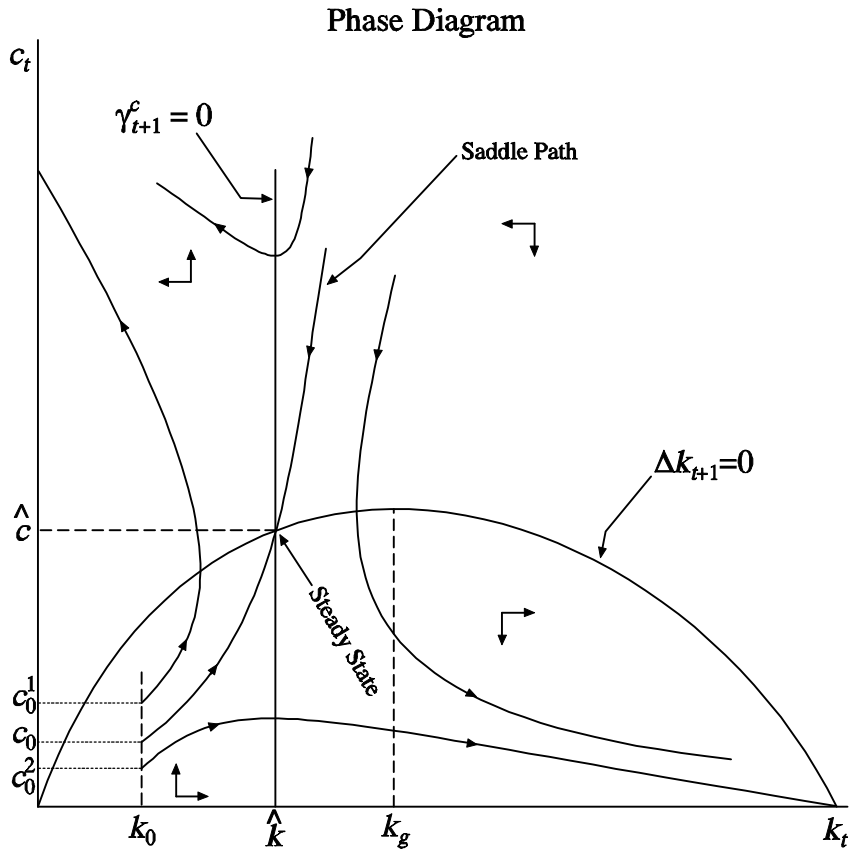
Suppose that the instantaneous utility function is isoelastic so that $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. The optimality condition can then be written as

$$\gamma^c_{t+1} = \left\{\frac{1+n}{1+\rho}[f'(k_{t+1}) + (1-\delta-n)]\right\}^{\frac{1}{\sigma}} - 1 \simeq \frac{1}{\sigma}[f'(k_{t+1}) - \rho - \delta]$$

so that $\gamma^c_{t+1} = 0$ if $k_{t+1} = \widehat{k}$ as this implies $f'(k_{t+1}) = \rho + \delta$. The $\gamma^c_{t+1} = 0$ locus is thus a vertical line at $k_{t+1} = \widehat{k}$ as shown on the diagram. For points to the left of this line $k_{t+1} < \widehat{k}$ so that $f'(k_{t+1}) > f'(\widehat{k})$ implying $f'(k_{t+1}) > \rho + \delta$ or $\gamma^c_{t+1} > 0$. This is indicated by the upward-pointing vertical arrows to the left of the $\gamma^c_{t+1} = 0$ line. Similarly, to the right of this line $\gamma^c_{t+1} < 0$ as indicated.

---

[13] To see this note that $U'(\widehat{c}) = \frac{1+n}{1+\rho}U'(\widehat{c})[f'(\widehat{k}) + (1-\delta-n)] \Rightarrow 1 + \rho = (1+n)[f'(\widehat{k}) + (1-\delta-n)]$ $\simeq f'(\widehat{k}) + (1-\delta-n) + n = 1 + f'(\widehat{k}) - \delta$.



The $\gamma^c_{t+1} = 0$ and $\Delta k_{t+1} = 0$ lines intersect at the steady-state values of c and k. The lines divide the (c, k) plane into four areas and the behavior of the system within each area is indicated by the arrows. For example, the ↑→ arrows indicate that in this region of the plane (to the left of the $\gamma^c_{t+1} = 0$ line and below the $\Delta k_{t+1} = 0$ line) the system tends to more in a northeasterly direction so that both c and k tend to increase. Note that, while this movement is in the general direction of the steady state, the same is not true in all of the areas defined by the two lines.

At any point on this diagram the motion of the system is described by the optimality condition and the law of motion for the capital stock. Thus, all of the paths are "optimal" in this regard. There is only one path - the "saddle path" - that leads to the steady state. Given the initial capital stock, k_0, the economy needs to choose an inital level of consumption equal to c_0 to get onto the saddle path. What makes the economy do this?

Suppose that initial level of consumption $c_0^1 > c_0$ were choosen. This would put the economy on the path shown with a declining rate of increase in the capital stock due to consumption being "too high" implying that saving is "too low". Eventually growth in the capital stock ceases (at the point where this path crosses the $\Delta k_{t+1} = 0$ line) and then the capital stock declines until it is zero (at the point where this path crosses the c axis). At this point output must also be zero and so consumption must drop to zero. This represents a violation of the optimality condition which implies a smooth path for consumption.

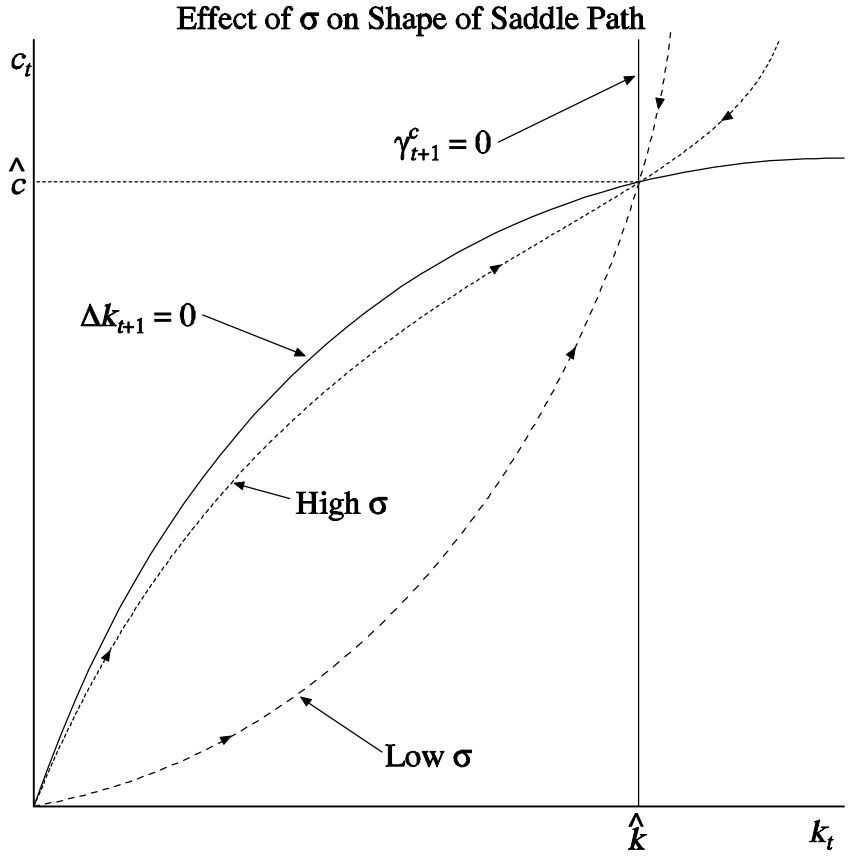
Suppose that initial level of consumption $c_0^2 < c_0$ were chosen. This would put the economy on the path shown with an increasing capital stock. This path represents, however, "oversaving" in the sense that the individuals would be able to enjoy higher consumption, if they were to destroy some of the capital stock.[14]

The shape of the saddle path and the rate of convergence to the steady state depend on the parameters of the model. I will take up this issue more generally once we have introduced technical progress into the model. For now I will concentrate on the role played by the aversion to fluctuations in consumption as measured by σ. Recall the interpretation of $\frac{1+n}{1+\rho}[f'(k_{t+1}) + (1-\delta-n)] \simeq f'(k_{t+1}) - \rho - \delta$ as the incentive to postpone consumption and the intertemporal elasticity of substitution, $\frac{1}{\sigma}$, as measuring the the responsiveness of households to this incentive. When σ is high, this response is small so consumption is high for each value of k and the saddle path lies close to the $\Delta k_{t+1} = 0$ line as shown on the diagram below. The households in this economy want to enjoy the high consumption available in the steady state, or something close to it, as soon as possible due to their aversion to fluctuations in their consumption stream.[15] When σ is low, households are will to tolerate the large fluctuaution in their consumption paths implied by having low consumption now and thus more rapid accumulation of capital and transition to the steady state where they will enjoy higher consumption. When $k < \widehat{k}$, which is the relevant case, the saddle path is below that for a high σ economy.

---

[14]This path violates the transversality condition for this problem. That condition is $\lim_{t\to\infty}\left(\frac{1+n}{1+\rho}\right)^t c_t^{-\sigma} k_t = 0$. The quantity $\left(\frac{1+n}{1+\rho}\right)^t c_t^{-\sigma}$ is the (present) value of the marginal unit of capital at time t so this condition requires the (present) value of the capital stock to go to zero as we go far into the future. To evaluate the condition observe that $c_t = c_0 \prod_{s=1}^{t}(1+\gamma_s^c)$ and recall $\gamma_{t+1}^c = \left\{\frac{1+n}{1+\rho}[f'(k_{t+1}) + (1-\delta-n)]\right\}^{\frac{1}{\sigma}} - 1$ so that $c_t = c_0 \prod_{s=1}^{t}\left(\left\{\frac{1+n}{1+\rho}[f'(k_s) + (1-\delta-n)]\right\}^{\frac{1}{\sigma}}\right)$ $= c_0\left(\frac{1+n}{1+\rho}\right)^{\frac{t}{\sigma}} \prod_{s=1}^{t}\left([f'(k_s) + (1-\delta-n)]^{\frac{1}{\sigma}}\right)$. Thus $c_t^{-\sigma} = c_0^{-\sigma}\left(\frac{1+\rho}{1+n}\right)^t \prod_{s=1}^{t}\left(\frac{1}{[f'(k_s)+(1-\delta-n)]}\right)$ so that $\left(\frac{1+n}{1+\rho}\right)^t c_t^{-\sigma} k_t = k_t c_0^{-\sigma} \prod_{s=1}^{t}\left(\frac{1}{[f'(k_s)+(1-\delta-n)]}\right)$. Along this path eventually $k_s > k_g$ implying $f'(k_s) < f'(k_g) = n + \delta$ so that $\frac{1}{[f'(k_s)+(1-\delta-n)]} > 1$ for all subsequent s. Thus $\lim_{t\to\infty} k_t c_0^{-\sigma} \prod_{s=1}^{t}\left(\frac{1}{[f'(k_s)+(1-\delta-n)]}\right) = \infty$ violating the transversality condition. Note that the saddle path satisfies this condition because $f'(\widehat{k}) = \rho + \delta > n + \delta$, by virture of the assumption that $\rho > n$, so that $\frac{1}{[f'(\widehat{k})+(1-\delta-n)]} < 1$.

[15]When we introduce tecnological progess below we shall see that σ plays a similar role in determining the steady state interest rate as does ρ. Thus the tendency for a high σ to result in behavior that looks like impatience should not be surprising.



Exogenous Technical Progress

So far, this model is incapable of generating sustained growth in per capita incomes. The assumption that $\lim_{k\to\infty} f'(k) \leq \delta + \rho$ ensures that, eventually, $f'(k) = \delta + \rho$ so that there is no longer any incentive to accumulate capital. One way to get per capita income to grow forever is to assume the existence of exogenous technical progress. We do this by letting E_t to be the efficiency of a unit of labor input at time t. E_t is assumed to grow at the exogenously given rate g so that $E_t = (1+g)^t E_0$. Total labor input is now measured in efficiency units and is given by $L_t E_t$ so that the production function is now written as $Y_t = F(K_t,\, L_t E_t)$. By defining $\tilde{y} = \frac{Y}{LE}$ and $\tilde{k} = \frac{K}{LE}$ to be the number of units of output and capital, respectively, per efficiency unit of labor input we can write this production function as $\tilde{y}_t = f(\tilde{k}_t)$.

The law of motion for $\tilde{k}$ is $\tilde{k}_{t+1} = (1-\delta-n-g)\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t$ where $\tilde{c}_t = \frac{C_t}{L_t E_t}$ is the amount of consumption per efficiency unit of labor input.

The trickiest part of this generalization of the model is the required modification to the objective function. Note that $\tilde{c}_t = \frac{c_t}{E_t}$ so that $c_t = \tilde{c}_t E_t = \tilde{c}_t (1+g)^t E_0$ where c_t is consumption per capita. It is the amount of consumption per person, c_t, that determines each individual's level of utility so lifetime preferences continue to be defined as before. We will, however, assume that the instantaneousc utility function is isoelastic so that $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$. To express utility in terms of consumption per unit of labor input use the

relationship between c_t and $\tilde{c}_t$ above to write $U(c_t) = \frac{(\tilde{c}_t(1+g)^t E_0)^{1-\sigma}-1}{1-\sigma} = \frac{\tilde{c}_t^{1-\sigma}(1+g)^{t(1-\sigma)}E_0^{1-\sigma}-1}{1-\sigma}$.

We use the law of motion for k to write $\tilde{c}_t = (1-\delta-n-g)k_t + f(k_t) - k_{t+1}$ which may be substituted into the utility function to give lifetime utility as

$$\sum_{t=0}^{\infty}\left(\frac{1+n}{1+\rho}\right)^t \frac{[(1-\delta-n-g)\tilde{k}_t + f(\tilde{k}_t) - \tilde{k}_{t+1}]^{1-\sigma}(1+g)^{t(1-\sigma)} - 1}{1-\sigma}$$

where we have set $E_0 = 1$ for simplicity. The social planner chooses a sequence of capital stocks $\left\{\tilde{k}_t\right\}_{t=1}^{\infty}$ given $\tilde{k}_0 > 0$ to maximize this quantity.

Taking the partial derivative with respect to $\tilde{k}_{t+1}$, setting it equal to zero, and doing a little algebra gives the optimality condition

$$\tilde{c}_t^{-\sigma} = \frac{(1+n)(1+g)^{1-\sigma}}{1+\rho}[f'(\tilde{k}_{t+1}) + (1-\delta-n-g)]\tilde{c}_{t+1}^{-\sigma}$$

which may be written as

$$(1+\gamma_{t+1}^{\tilde{c}})^{\sigma} = \frac{(1+n)(1+g)^{1-\sigma}}{1+\rho}[f'(\tilde{k}_{t+1}) + (1-\delta-n-g)]$$

where $\gamma_{t+1}^{\tilde{c}} = \frac{\tilde{c}_{t+1}}{\tilde{c}_t} - 1$, the growth rate of $\tilde{c}$.

Imposing the steady-state condition $\gamma_{t+1}^{\tilde{c}} = 0$ yields the condition

$$1+\rho = [f'(\widehat{\tilde{k}}) + (1-\delta-n-g)](1+n)(1+g)^{1-\sigma}$$

which implies $f'(\widehat{\tilde{k}}) - \delta = \rho + \sigma g$.[16] This condition showns that, in the steady state, the net marginal product of capital is equal to $\rho + \sigma g$, a quantity that may be refered to as the "effective discount rate". Economies with higher values of σ will thus have higher effective discount rates and lower steady-state amounts of capital per unit of labor input (and hence per worker). Recall that σ measures a consumer's aversion to fluctuations in their consumption over time.

The other condition becomes $\widehat{\tilde{c}} = f(\widehat{\tilde{k}}) - (n+g+\delta)\widehat{\tilde{k}}$. Setting the derivative this expression with respect to $\widehat{\tilde{k}}$ equal to zero shows that the capital stock maximizing per capita consumption is the solution to $f'(\tilde{k}) = n+g+\delta < \rho+\sigma g+\delta = f'(\widehat{\tilde{k}})$ provided

[16] To see this note that $(1+g)^{1-\sigma} \simeq 1+g(1-\sigma)$ so that the steady-state condition becomes $1+\rho \simeq [f'(\widehat{\tilde{k}}) + (1-\delta-n-g)](1+n)[1+g(1-\sigma)] \simeq f'(\widehat{\tilde{k}}) + (1-\delta-n-g) + n + g(1-\sigma) = 1 + f'(\widehat{\tilde{k}}) - \delta - \sigma g$.

$\rho > n + (1-\sigma)g$, a condition we will impose so, as before, the steady-state capital stock is less than that maximizing consumption per capita.[17]

In the steady state $\tilde{k}$, $\tilde{y}$, and $\tilde{c}$ are constant so k, y, and c all grow at the rate g. Note that this is precisely the same conclusion as that from the Solow growth model.

Exercises:

1. Verify that the production function $y = Ak^{\alpha}$ satisfies the assumptions we have made about production functions.

2. Consider the production function $y = Ak^{\alpha}$. For the cases of $\alpha = .3$ and $\alpha = .5$ draw, on the same chart, diagrams equivalent to that labeled "Unique Steady State" in the text using the parameter values $A = 1$, $\rho = .025$, and $\delta = .04$. Compute the steady state output levels in each case. Describe the differences in the two cases.

3. Show that, in the steady state of the model with constant returns to scale and exogenous technical progress, the real interest rate is constant and the real wage grows at the rate of technical progress.

4. In this problem you will solve a version of the optimal growth model without exogenous technical progress numerically. We will work with a version of the model with $f(k) = Ak^{\alpha}$ and $\sigma = 1$ so that the law of motion becomes $k_{t+1} = Ak_t^{\alpha} + (1 - n - \delta)k_t - c_t$ while the optimality condition can be written as $c_{t+1} = \left\{\frac{1+n}{1+\rho}[\alpha A k_{t+1}^{\alpha-1} + (1-\delta-n)]\right\}c_t$. So, begining with k_0 and c_0 we can use the law of motion to calculate k_1, which, with c_0, can be used in the optimality condition to calculate c_1. We can continue in this way until, if we have the correct value of c_0, we get to the steady state. If we don't have the correct value we will encounter the difficulties described above. We will use the parameter values $A = 1$, $\alpha = .3$, $\rho = .025$, $\delta = .04$, $n = .01$ and begin with a capital stock of $k_0 = 1$.

(i) Calculate the steady state values of k, c, and y.

(ii) Calculate the golden rule values of k, c, and y.

(iii) Set $c_0 = .4$. Calculate and graph the paths of k, c, and y for $t = 0, \ldots, 100$. Describe what happens.

(iv) Set $c_0 = .55$Calculate and graph the paths of k, c, and y for $t = 0, \ldots, 100$. Describe what happens.

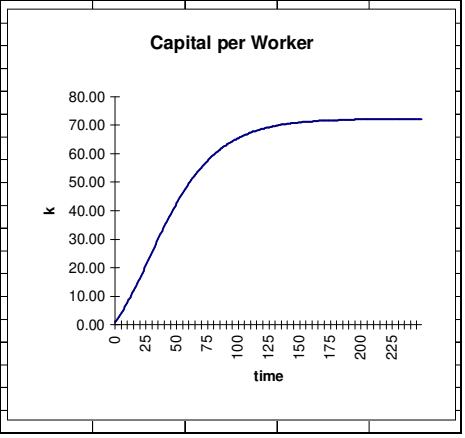
(v) Set $c_0 = .516903720370562$. Calculate and graph the paths of k, c, and y for $t = 0, \ldots, 100$. Describe what happens. Graph the growth rate of y and the real interest rate on the same chart for $t = 0, \ldots, 100$. Describe what you see. Graph the ratio of consumption to output for $t = 0, \ldots, 100$. Describe what you see.

---

[17]This condition is also necssary to ensure that the objective function is bounded. The term mutiplying $\tilde{c}_t^{1-\sigma}$ is $\left(\frac{(1+n)(1+g)^{1-\sigma}}{1+\rho}\right)^t \simeq \left(\frac{(1+n)[1+g(1-\sigma)]}{1+\rho}\right)^t$ which is finite only if $\frac{(1+n)[1+g(1-\sigma)]}{1+\rho} < 1$.

HINT: I recommend that you do not attempt these calculations by hand or even with a calculator. I used a spreadsheet program (to be specific, MS Excel) and set up the page like this:

alpha=	0.3	t	kt	yt	ct	Growth Rate
		0	1.000	1.000	0.400	
A=	1	1	1.550	1.141	0.461	14.05
rho=	0.025	2	2.152	1.258	0.512	10.34
delta=	0.04	3	2.791	1.361	0.553	8.12
n=	0.01	4	3.459	1.451	0.586	6.65
		5	4.151	1.533	0.613	5.62
k0=	1	6	4.864	1.607	0.633	4.87
c0=	0.4	7	5.594	1.676	0.649	4.29
y0=	1	8	6.342	1.740	0.660	3.83
kss=	8.918414	9	7.105	1.801	0.667	3.47
yss=	1.927908	10	7.884	1.858	0.671	3.17
css=	1.481987	11	8.676	1.912	0.672	2.92
		12	9.482	1.964	0.670	2.70
		13	10.302	2.013	0.666	2.52
		14	11.134	2.061	0.660	2.36
		15	11.978	2.106	0.652	2.22
		16	12.833	2.150	0.643	2.09
		17	13.699	2.193	0.632	1.98
		18	14.575	2.234	0.620	1.88



You will need to be careful to properly deal with the possibility of negative values for k. Once the spreadsheet is set up, which takes a little bit of programming, you can just change the c_0 value and all the numbers and graphs will change as appropriate. Print the graphs and then go on to the next value for c_0. If you are unsure of how to do this see me for help.

5. One way to study the effects of an increased desire or incentive to save on growth is to consider a decrease in ρ. Show that, while the steady-state levels of consumption and output per capita are increased but such a change, the steady-state growth rates are unaffected. Show also, that the level of consumption immediately following the decrease in ρ is lower than that prior to the decrease. HINT: Use the phase diagram. Begin with the economy in a steady state. Consider the effects of a decrease in ρ on the positions and shapes of the $\gamma_{t+1}^{c} = 0$ and $\Delta k_{t+1} = 0$ lines. Observe that when these lines shift the saddle path also shifts and remember that the economy is always on the saddle path.

6. Show that, in the steady-state, the saving rate (gross investment as fraction of GDP) is less than capital's share of output (total payments to capital as a fraction of GDP).

7. Consider the model without exogenous technical progress. Suppose that the production function is Cobb-Douglas so that $y_t = Ak_t^{\alpha}$.

(i) Show that the steady-state saving rate is given by $\frac{\alpha(n+\delta)}{\rho+\delta}$.

(ii) Show that this saving rate is indeed a fraction.

(iii) If we define $z_t = \frac{c_t}{y_t}$, it can be shown that the rate of change of z_t is given by $\frac{\Delta z_{t+1}}{z_t} = \alpha A k^{\alpha-1}\left(\frac{1}{\sigma} - s_t\right) - (\rho+\delta)\left(\frac{1}{\sigma} - \widehat{s}\right)$ where $s_t = 1 - \frac{c_t}{y_t}$ is the saving rate and $\widehat{s}$ is its steady-state value. Use this result to show that if $\widehat{s} = \frac{1}{\sigma}$ then $s_t = \frac{1}{\sigma}$ for all t.

(iv) Combine the results in parts (i) and (iii) to argue that, if $\sigma\alpha(n+\delta) = \rho+\delta$, the saving rate is constant as the economy transitions to the steady state.

HINTS: In (i) observe that the saving rate is equal to $1 - \frac{c_t}{y_t}$ and use the conditions defining the steady state. In (iii) show that if $s_t > \frac{1}{\sigma}$ the implied behavior of z_t is inconsistent with the assumption that $\widehat{s} = \frac{1}{\sigma}$.